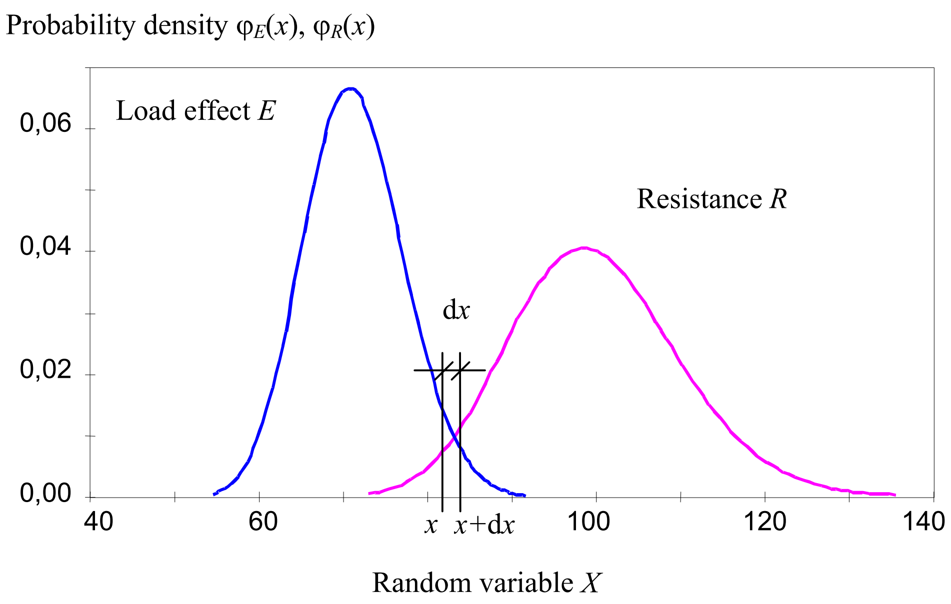
IMPLEMENTATION OF EUROCODES

HANDBOOK 2

RELIABILITY BACKGROUNDS



Guide to the basis of structural reliability and risk engineering related to Eurocodes, supplemented by practical examples

LEONARDO DA VINCI PILOT PROJECT CZ/02/B/F/PP-134007

DEVELOPMENT OF SKILLS FACILITATING IMPLEMENTATION OF EUROCODES

Leonardo da Vinci Pilot Project CZ/02/B/F/PP-134007

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HANDBOOK 2

RELIABILITY BACKGROUNDS

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DEVELOPMENT OF SKILLS FACILITATING IMPLEMENTATION OF EUROCODES

HANDBOOK 2

BASIS OF STRUCTURAL RELIABILITY AND RISK ENGINEERING

FOREWORD

The Leonardo da Vinci Pilot Project CZ/02/B/F/PP-134007, "Development of Skills Facilitating Implementation of Structural Eurocodes" addresses the urgent need to implement the new system of European documents related to design of construction works and products. These documents, called Eurocodes, are systematically based on recently developed Council Directive 89/106/EEC "The Construction Products Directive" and its Interpretative Documents ID1 and ID2. Implementation of Eurocodes in each Member State is a demanding task as each country has its own long-term tradition in design and construction.

The project should enable an effective implementation and application of the new methods for designing and verification of buildings and civil engineering works in all the partner countries (CZ, DE, ES, IT, NL, SI, UK) and in other Member States. The need to explain and effectively use the latest principles specified in European standards is apparent from various enterprises, undertakings and public national authorities involved in construction industry and also from universities and colleges. Training materials, manuals and software programmes for education are urgently required.

The submitted Handbook 2 is one of 5 upcoming handbooks intended to provide required manuals and software products for training, education and effective implementation of Eurocodes:

Handbook 1: Basis of Structural Design
Handbook 2: Reliability Backgrounds
Handbook 3: Load Effect for Buildings
Handbook 4: Load Effect for Bridges
Handbook 5: Design of Buildings for Fire Situation

It is expected that the Handbooks will address the following intents in further harmonisation of European construction industry:

- reliability improvement and unification of the process of design;
- development of the single market for products and for construction services;
- new opportunities for the trained primary target groups in the labour market.

The Handbook 2 is focused on the basis of structural reliability and risk engineering related to Eurocodes. The following topics are treated in particular:

- basic concepts of structural reliability;
- elementary methods of the reliability theory;
- reliability differentiation and design working life;
- design assisted by testing;
- assessment of existing structures;
- basis of risk assessment.

Annex A to the Handbook 2 provides a review of "Basic Statistical Concepts and Techniques" frequently used in the text. Annex B provides an extension of elementary methods of structural reliability and annex C describes calibration procedures that may be used for specification of reliability elements. The Handbook 2 is written in a user-friendly way employing only basic mathematical tools. Attached software products accompanying a number of examples enable applications of general rules in practice.

A wide range of potential users of the Handbooks and other training materials includes practising engineers, designers, technicians, experts of public authorities, young people - high school and university students. The target groups come from all territorial regions of the partner countries. However, the dissemination of the project results is foreseen to be spread into all Member States of CEN and other interested countries.

Prague 05/2005

CHAPTER I - BASIC CONCEPTS OF STRUCTURAL RELIABILITY

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Summary

Uncertainties affecting structural performance can never be entirely eliminated and must be taken into account when designing any construction work. Various design methods and operational techniques for verification of structural reliability have been developed and worldwide accepted in the past. The most advanced operational method of partial factors is based on probabilistic concepts of structural reliability and available experience. General principles of structural reliability can be used to specify and further calibrate partial factors and other reliability elements. Moreover, developed calculation procedures and convenient software products can be used directly for verification of structural reliability using probabilistic concepts and available experimental data.

1 INTRODUCTION

1.1 Background materials

Basic concepts of structural reliability are codified in a number of national standards, in the new European document EN 1990 [1] and the International Standard ISO 2394 [2]. Additional information may be found in the background document developed by JCSS [3] and in recently published handbook to EN 1990 [4]. Guidance for application of probabilistic methods of structural reliability may be found in working materials provided by JCSS [5] and in relevant literature listed in [4 and [5]. Elementary methods of the theory of reliability are described in Chapter II and III in this Handbook 2.

1.2 General principles

General principles of structural reliability are described in both the international documents EN 1990 [1] and ISO 2394 [2]. Basic requirements on structures are specified in Section 2 of EN 1990 [1]: a structure shall be designed and executed in such a way that it will, during its intended life, with appropriate degrees of reliability and in an economic way

- sustain all actions and influences likely to occur during execution and use;
- remain fit for the use for which it is required.

It should be noted that two aspects are explicitly mentioned: reliability and economy (see also Handbook 1). However, in this Handbook 2 we shall be primarily concern with reliability of structures, which include

- structural resistance;
- serviceability;
- durability.

Additional requirements may concern fire safety of structures (see Handbook 5) or other accidental design situations. In particular it is required by EN 1990 [1] that in the case of fire, the structural resistance shall be adequate for the required period of time.

To verify all the aspects of structural reliability implied by the above-mentioned basic requirements, an appropriate design lifetime, design situations and limit states should be considered (as described in Handbook 1). Note that the basic lifetime for a common building is 50 years and that, in general, four design situations are identified: permanent, transient, accidental and seismic. Two types of limit states are normally verified: ultimate limit states and serviceability limit states. Detail guidance is provided in Handbook 1.

2 UNCERTAINTIES

2.1 Classification of uncertainties

It is well recognised that construction works are complicated technical systems suffering from a number of significant uncertainties in all stages of execution and use. Depending on the nature of a structure, environmental conditions and applied actions, various types of uncertainties become more significant than the others. The following types of uncertainties can be identified in general:

- natural randomness of actions, material properties and geometric data;
- statistical uncertainties due to a limited size of available data;
- uncertainties of the resistance and load effect models due to simplifications of actual conditions;
- vagueness due to inaccurate definitions of performance requirements;
- gross errors in design, during execution and use;
- lack of knowledge concerning behaviour of new materials and actions in actual conditions.

The order of the listed uncertainties corresponds approximately to the decreasing level of current knowledge and available theoretical tools for their description and consideration in design (see following sections). It should be emphasized that most of the above listed uncertainties (randomness, statistical and model uncertainties) can never be eliminated absolutely and must be taken into account when designing any construction work.

2.2 Available tools to describe uncertainties

Natural randomness and statistical uncertainties may be relatively well described by available methods provided by the theory of probability and mathematical statistics. In fact the EN 1990 [1] gives some guidance on available techniques. However, lack of credible experimental data (e.g. for new materials, some actions including environmental influences and also for some geometrical properties) causes significant problems. In some cases the available data are inhomogeneous, obtained under different conditions (e.g. for material resistance, imposed loads, environmental influences, for inner dimensions of reinforced concrete cross-sections). Then it may be difficult, if not impossible, to analyse and use them in design.

The uncertainties of computational models may be to a certain extent assessed on the basis of theoretical and experimental research. EN 1990 [1] and materials of JCSS [5] provide some guidance. The vaguenesses caused by inaccurate definitions (in particular of serviceability and other performance requirements) may be partially described by the means of the theory of fuzzy sets. However, these methods have a little practical significance, as suitable experimental data are rarely available. The knowledge of the behaviour of new materials and structures may be gradually increased through theoretical analyses verified by experimental research.

The lack of available theoretical tools is obvious in the case of gross errors and lack of knowledge, which are nevertheless often the decisive causes of structural failures. To limit gross errors due to human activity, a quality management system including the methods of statistical inspection and control may be effectively applied.

Various design methods and operational techniques, which take these uncertainties into account, have been developed and worldwide used. The theory of structural reliability provides background concept techniques and theoretical bases for description and analysis of the above-mentioned uncertainties concerning structural reliability.

3 RELIABILITY

3.1 General

The term "reliability" is often used very vaguely and deserves some clarification. Often the concept of reliability is conceived in an absolute (black and white) way – the structure either is or isn't reliable. In accordance with this approach the positive statement is understood in the sense that "a failure of the structure will never occur". This interpretation is unfortunately an oversimplification. Although it may be unpleasant and for many people perhaps unacceptable, the hypothetical area of "absolute reliability" for most structures (apart from exceptional cases) simply does not exist. Generally speaking, any structure may fail (although with a small or negligible probability) even when it is declared as reliable.

The interpretation of the complementary (negative) statement is usually understood more correctly: failures are accepted as a part of the real world and the probability or frequency of their occurrence is then discussed. In fact in the design it is necessary to admit a certain small probability that a failure may occur within the intended life of the structure. Otherwise designing of civil structures would not be possible at all. What is then the correct interpretation of the keyword "reliability" and what sense does the generally used statement "the structure is reliable or safe" have?

3.2 Definition of reliability

A number of definitions of the term "reliability" are used in literature and in national and international documents. ISO 2394 [2] provides a definition of reliability, which is similar to the approach of national standards used in some European countries: reliability is the ability of a structure to comply with given requirements under specified conditions during the intended life, for which it was designed. In quantitative sense reliability may be defined as the complement of the probability of failure.

Note that the above definition of reliability includes four important elements:

- given (performance) requirements – definition of the structural failure,
- time period – assessment of the required service-life T,
- reliability level – assessment of the probability of failure P_f,
- conditions of use – limiting input uncertainties.

An accurate determination of performance requirements and thus an accurate specification of the term failure are of uttermost importance. In many cases, when considering the requirements for stability and collapse of a structure, the specification of the failure is not very complicated. In many other cases, in particular when dealing with various requirements of occupants' comfort, appearance and characteristics of the environment, the appropriate definitions of failure are dependent on several vaguenesses and inaccuracies. The transformation of these occupants' requirements into appropriate technical quantities and precise criteria is very hard and often leads to considerably different conditions.

In the following the term failure is being used in a very general sense denoting simply any undesirable state of a structure (e.g. collapse or excessive deformation), which is unambiguously given by structural conditions.

The same definition as in ISO 2394 is provided in Eurocode EN 1990 [1] including note that the reliability covers the load-bearing capacity, serviceability as well as the durability of a structure. Fundamental requirements include the statement (as already mentioned) that "a structure shall be designed and executed in such a way that it will, during its intended life with appropriate degrees of reliability and in an economic way sustain all actions and influences likely to occur during execution and use, and remain fit for the use for which it is required". Generally a different level of reliability for load-bearing capacity and for serviceability may be accepted for a structure or its parts. In the documents [1] and [2] the probability of failure P_f (and reliability index β) are indicated with regard to failure consequences (see Handbook 1).

3.3 Probability of failure

The most important term used above (and in the theory of structural reliability) is evidently the probability of failure P_f. In order to defined P_f properly it is assumed that structural behaviour may be described by a set of basic variables $\boldsymbol{X} = [X_1, X_2, \dots , X_n]$ characterizing actions, mechanical properties, geometrical data and model uncertainties. Furthermore it is assumed that the limit state (ultimate, serviceability, durability or fatigue) of a structure is defined by the limit state function (or the performance function), usually written in an implicit form as

$$\mathrm{Z}(\boldsymbol{X}) = 0 \tag{1}$$

The limit state function $\mathrm{Z}(\boldsymbol{X})$ should be defined in such a way that for a favourable (safe) state of a structure the function is positive, $\mathrm{Z}(\boldsymbol{X}) \geq 0$, and for a unfavourable state (failure) of the structure the limit state function is negative, $Z(\boldsymbol{X}) < 0$ (a more detailed explanation is given in the following Chapters of this Handbook 2).

For most limit states (ultimate, serviceability, durability and fatigue) the probability of failure can be expressed as

$$P_f = \mathrm{P}\{\mathrm{Z}(\boldsymbol{X}) < 0\} \tag{2}$$

The failure probability P_f can be assessed if basic variables $\boldsymbol{X} = [X_1, X_2, \dots , X_n]$ are described by appropriate probabilistic (numerical or analytical) models. Assuming that the basic variables $\boldsymbol{X} = [X_1, X_2, \dots , X_n]$ are described by time independent joint probability density function $\varphi_{\boldsymbol{X}}(\boldsymbol{x})$ then the probability P_f can be determined using the integral

$$P_f = \int_{\mathrm{Z}(X)<0} \varphi_X(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{3}$$

More complicated procedures need to be used when some of the basic variables are time-dependent. Some details concerning theoretical models for time-dependent quantities (mainly actions) and their use for the structural reliability analysis are given in other Chapters of this Handbook 2. However, in many cases the problem may be transformed to a time-independent one, for example by considering in equation (2) or (3) a minimum of the function $\mathrm{Z}(\boldsymbol{X})$ over the reference period T.

Note that a number of different methods [2] and software products [7, 8, 10] are available to calculate failure probability P_f defined by equation (2) or (3).

3.4 Reliability index

An equivalent term to the failure probability is the reliability index β, formally defined as a negative value of a standardized normal variable corresponding to the probability of failure P_f. Thus, the following relationship may be considered as a definition

$$\beta = -\Phi_U^{-1}(P_f) \tag{4}$$

Here $\Phi_u^{-1}(p_f)$ denotes the inverse standardised normal distribution function. At present the reliability index β defined by equation (4) is a commonly used measure of structural reliability in several international documents [1], [2], [5].

It should be emphasized that the failure probability P_f and the reliability index β represent fully equivalent reliability measures with one to one mutual correspondence given by equation (4) and numerically illustrated in Table 1.

Table 1. Relationship between the failure probability P_f and the reliability index β.

P_f	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
β	1,3	2,3	3,1	3,7	4,2	4,7	5,2

In EN 1990 [1] and ISO 2394 [2] the basic recommendation concerning a required reliability level is often formulated in terms of the reliability index β related to a certain design working life.

3.5 Time variance of failure probability

When the vector of basic variables $\boldsymbol{X} = X_1, X_2, \dots, X_m$ is time variant, then the failure probability p is also time variant and should be always related to a certain reference period T, which may be generally different from the design working life T_d. Considering a structure of a given reliability level, the design failure probability $p_d = p_n$ related to a general reference period $T_n = n\ T_1$ can be derived from the alternative probability $p_a = p_1$ corresponding to $T_a = T_1$ (to simplify notation note that the previously used subscript "d" corresponds now to "n" and subscript "a" to "1"). Detail description of this transformation is provided in Chapter III.

4 DESIGN TARGETS

4.1 Indicative values of design working life

Design working life T_d is an assumed period of time for which a structure or part of it is to be used for its intended purpose with anticipated maintenance but without major repair being necessary. In the recent documents CEN [1] and ISO [2] indicative values of T_d are provided for five categories of structures as shown in Table 2.

A more detailed specification of structural categories and design working lives may be found in some national standards. In general the design working lives may be greater (in some cases by 100 %) than those given in Table 2. For example the design working life for temporary structures may be 15 years, for agricultural structures 50 years, for apartment and office buildings 100 years, and for railways structures, dams, tunnels and other underground engineering works 120 years or more.

Table 2. Indicative design working life T_d.

Category	Design working life T_d (years)	Examples
1	10	Temporary structures
2	10 to 25	Replaceable structural parts, bearings, girders
3	15 to 30	Agricultural and similar structures
4	50	Building structures and common structures
5	100 and more	Monumental building or civil structures, bridges

4.2 Target reliability level

Design failure probabilities p_d are usually indicated in relation to the expected social and economical consequences. EN 1990 [1] provides the classification of target reliability levels into three classes of consequences (high, normal, low) and indicates the adequate reliability indexes β for two reference periods T (1 year and 50 years). No explicit link to the design working life T_d. is given Similar β-values may be found also in some national standards and international standards ISO [2]. Detail description of the target is given in Chapter III in this Handbook 2.

It should be underlined that the couple of β values (β_a and β_d) recommended in [1] for each reliability class (for 1 year and 50 years) correspond to the same reliability level. Practical application of these values depend on the reference period T_a considered in the verification, which may be connected with available information concerning time variant vector of basic variables $\boldsymbol{X} = X_1, X_2, ..., X_n$. For example, if the reliability class 2 and 50 years design working period is considered, then the reliability index β_d = 3,8 should be used in the verification of structural reliability. The same reliability level corresponding to the class 2 is achieved when the time period T_a = 1 years and β_a = 4,7 are considered. Thus, various reference periods T_a, in general different from the design working life T_d, may be used for achieving a certain reliability level.

5 DESIGN METHODS IN PRACTICE

5.1 General

During their historical development the design methods have been closely linked to the available empirical, experimental as well as theoretical knowledge of mechanics and the theory of probability. The development of various empirical methods for structural design gradually crystallized in the twentieth century in three generally used methods, which are, in various modifications, still applied in standards for structural design until today: the permissible stresses method, the global factor and partial factor methods. All these methods are often discussed and sometimes reviewed or updated.

The following short review of historical development illustrates general formats of above mentioned design methods, indicate relevant measures that are applied to take into account various uncertainties of basic variables and to control resulting structural reliability. In addition a short description of probabilistic methods of structural reliability and their role in further development of design procedures is provided. Detailed description of probabilistic methods of structural reliability is given in Chapter II, Chapter III and in Annex B of this Handbook 2.

5.2 Permissible stresses

The first of the worldwide-accepted design methods for structural design is the method of permissible stresses that is based on linear elasticity theory. The basic design condition of this method can be written in the form

$$\sigma_{\max} < \sigma_{\text{per}}, \text{ where } \sigma_{\text{per}} = \sigma_{\text{crit}} / k \tag{5}$$

The coefficient k (greater than 1) is the only explicit measure supposed to take into account all types of uncertainties (some implicit measures may be hidden). Moreover, only a local effect (a stress) $\sigma_{\max}$ is compared with the permissible stress σ_{per} and, therefore, a local (elastic) behaviour of a structure is used to guarantee its reliability. No proper way is provided for treating geometric non-linearity, stress distribution and ductility of structural materials and members. For that reasons the permissible stress method leads usually to conservative and uneconomical design.

However, the main insufficiency of the permissible stress method is lack of possibility to consider uncertainties of individual basic variables and computational models used to assess load effects and structural resistances. Consequently, reliability level of structures exposed to different actions and made of different material may be not only conservative (uneconomical) but also considerably different.

5.3 Global safety factor

The second widespread method of structural design is the method of global safety factor. Essentially it is based on a condition relating the standard or nominal values of the structural resistance R and load effect E. It may be written as

$$s = R / E > s_0 \tag{6}$$

Thus the calculated safety factor s must be greater than its specified value s_0 (for example s_0 = 1,9 is commonly required for bending resistance of reinforced concrete members). The global safety factor method attempts to take into account realistic assumptions concerning structural behaviour of members and their cross-sections, geometric non-linearity, stress distribution and ductility; in particular through the resulting quantities of structural resistance R and action effect E.

However, as in the case of the permissible stresses method the main insufficiency of this method remains a lack of possibility to consider the uncertainties of particular basic quantities and theoretical models. The probability of failure can, again, be controlled by one explicit quantity only, by the global safety factor s. Obviously harmonisation of reliability degree of different structural members made of different materials is limited.

5.4 Partial factor method

At present, the most advanced operational method of structural design [1, 2] accepts the partial factor format (sometimes incorrectly called the limit states method) usually applied in conjunction with the concept of limit states (ultimate, serviceability or fatigue). This method can be generally characterised by the inequality

$$E_d\,(F_d, f_d, a_d, \theta_d) < R_d\,(F_d, f_d, a_d, \theta_d) \tag{7}$$

where the design values of action effect E_d and structural resistance R_d are assessed considering the design values of basic variables describing the actions $F_d = \psi\, \gamma_F\, F_k$, material properties $f_d = f_k / \gamma_m$, dimensions $a_d + \Delta a$ and model uncertainties θ_d. The design values of these quantities are determined (taking into account various uncertainties) using their characteristic values (F_k, f_k, a_k, θ_k), partial factors γ, reduction factors ψ and other measures of reliability [1, 2, 3, 4], Thus the whole system of partial factors and other reliability elements may be used to control the level of structural reliability. Detailed description of the partial factor methods used in Eurocodes method is provided in Handbook 1.

Compared with previous design methods the partial factor format obviously offers the greatest possibility to harmonise reliability of various types of structures made of different materials. Note, however, that in any of the above listed design methods the failure probability is not applied directly. Consequently, the failure probability of different structures made of different materials may still considerably vary even though sophisticated calibration procedures were applied. Further desired calibrations of reliability elements on probabilistic bases are needed; it can be done using the guidance provided in the International standard ISO 2394 [2] and European document EN 1990 [1].

5.5 Probabilistic methods

The probabilistic design methods introduced in the International Standard [2] are based on a requirement that during the service life of a structure T the probability of failure P_f does not exceed the design value p_d or the reliability index β is greater than its design value β_d

$$P_f \leq P_d \text{ or } \beta > \beta_d \tag{8}$$

In EN 1990 [1] the basic recommended reliability index for ultimate limit states β_d = 3,8 corresponds to the design failure probability $P_d = 7{,}2 \times 10^{-5}$, for serviceability limit states $\beta_d = 1{,}5$ corresponds to $P_d = 6{,}7 \times 10^{-2}$. These values are related to the design working life of 50 years that is considered for building structures and common structures. In general greater β - values should be used when a short reference period (one or five years) will be used for verification of structural reliability.

It should be mentioned that probabilistic methods are not yet commonly used in design praxis. However, the developed calculation procedures and software products (for example [7, 8] and [10]) already enable the direct verification of structural reliability using probabilistic concepts and available experimental data. Recently developed software product CodeCal [10] is primarily intended for calibration of codes based on the partial factor method.

In Chapter II of this Handbook 2 numerical examples will be presented to illustrate the methods discussed above.

6 DESIGN ASSISTED BY TESTING

In some cases there is a need to base the design on a combination of tests and calculations, for instance if no adequate calculation model is available. The tests may vary from wind tunnel tests to prototype testing of new structural materials, elements or assemblies. Tests may also be carried out during or after execution to confirm the design assumptions. The extreme example is a proof load. For design by testing the following types of tests can be distinguished:

a) tests to establish directly the resistance for given loading conditions
b) tests to obtain specific material properties
c) tests to reduce model uncertainties in loads, load effects or resistance models

Test should be set up and evaluated in such a way that the usual required level of reliability is achieved. The derivation of a characteristic or design value should take into account the scatter of test data, statistical uncertainty associated with the number of tests and prior statistical knowledge. If the response of the structure or structural member or the resistance of the material depends on influences not sufficiently covered by the tests such as duration or scale effects, corrections should be made.

When evaluating test results, the behaviour of test specimens and failure modes should be compared with theoretical predictions. When significant deviations from a prediction occur, an explanation should be sought: this might involve additional testing, perhaps under different conditions, or modification of the theoretical model.

The evaluation of test results should be based on statistical methods. In Eurocodes both Bayesian and classical frequentistic methods are used. When frequentistic methods are used a confidence level has to be chosen. The level of the confidence interval may influence the final value. On the average, a confidence level of 0.75 leads to the same result as the Bayesian methods. For this reason 0.75 is chosen in most cases, however also other numbers are used (eg. 0.85 in EN 1995).

In Basis of Design the preference is given for Bayesian methods, which generally is believed to be more consistent with modern reliability theory then frequentistic methods. Moreover, Bayesian methods provide a formal framework for the use of prior knowledge, which is essential especially in the case of small samples and quality control methods. In most Eurocodes special rules for small samples are presented, but in general without formal background.

Rules for execution and evaluation of design by testing are presented in Annex D of EN 1990 Basis of Design. For detailed background and worked examples, the reader is referred to Annex A of this Handbook 2.

7 CONCLUDING REMARKS

The basic concepts of the probabilistic theory of reliability are characterized by two equivalent terms, the probability of failure P_f and the reliability index β. Although they provide limited information on the actual frequency of failures, they remain the most important and commonly used measures of structural reliability. Using these measures the theory of structural reliability may be effectively applied for further harmonisation of reliability elements and for extensions of the general methodology for new, innovative structures and materials.

Historical review of the design methods worldwide accepted for verification of structural members indicates different approaches to considering uncertainties of basic variables and computational models. The permissible stresses method proves to be rather conservative (and uneconomical). The global safety factor and partial factor methods lead to similar results. Obviously, the partial factor method, accepted in the recent EN documents, represents the most advanced design format leading to a suitable reliability level that is relatively close to the level recommended in EN 1990 (β = 3,8). The most important advantage of the partial factor method is the possibility to take into account uncertainty of individual basic variables by adjusting (calibrating) the relevant partial factors and other reliability elements.

Various reliability measures (characteristic values, partial and reduction factors) in the new structural design codes using the partial factor format are partly based on probabilistic methods of structural reliability, partly (to a great extent) on past empirical experiences. Obviously the past experience depends on local conditions concerning climatic actions and traditionally used construction materials. These aspects may be considerably different in different countries. That is why a number of reliability elements and parameters in the present suite of European standards are open for national choice.

It appears that further harmonisation of current design methods will be based on calibration procedures, optimisation methods and other rational approaches including the use of methods of the theory of probability, mathematical statistics and the theory of reliability. The probabilistic methods of structural reliability provide the most important tool for gradual improvement and harmonisation of the partial factor method for various structures from different materials. Moreover, developed software products enable direct application of reliability methods for verification of structures using probabilistic concepts and available data.

Design assisted by testing may be used when there is a need to base the design on a combination of tests and calculations. The tests may vary from wind tunnel tests to prototype testing of new structural materials, elements or assemblies. Tests may also be carried out during or after execution to confirm the design assumptions. An operational technique recommended in [1] is described in Chapter IV of this Handbook 2.

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ATTACHMENTS

1. MATHCAD sheet “Beta-Time.mcd”

Mathcad sheet "Beta-Time" is intended for transformation of probability and reliability index Beta" for different reference periods.

Attachment 1 - MATHCAD sheet "Beta-Time.mcd"

Mathcad sheet Beta -Time

Mathcad sheet "Beta-Time" is intended for transformation of probability and reliability index Beta" for different reference periods"

1 Input data $n := 10, 20 .. 100$ $\beta1 := 0, 1 .. 6$

2 Probability

$$p1(\beta1) := pnorm(-\beta1, 0, 1) \qquad pn(\beta1, n) := \left[1 - (1 - p1(\beta1))^n\right]$$

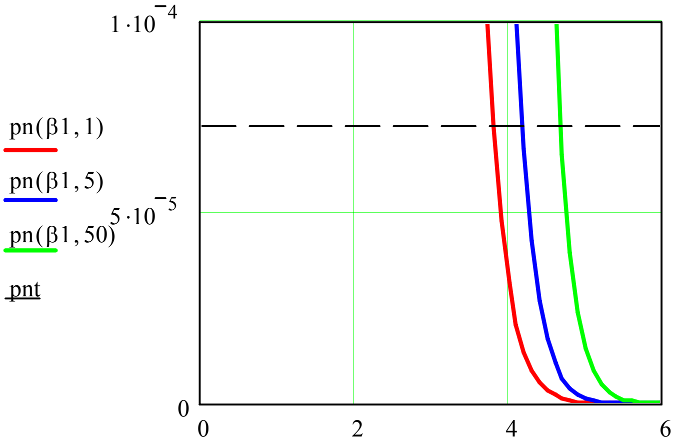
3 Reliability index

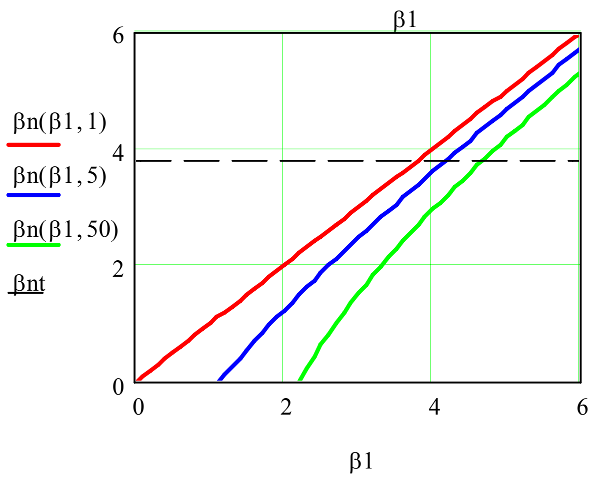
$$\beta n(\beta1, n) := -qnorm\left[\left[1 - (1 - pnorm(-\beta1, 0, 1))^n\right], 0, 1\right] \qquad \beta n(4.7, 50) = 3.826$$

4 Numerical results

$\beta1 =$	$p1(\beta1) =$	$pn(\beta1, 50) =$
0	0.5	1
1	0.159	1
2	0.023	0.684
3	$1.35 \cdot 10^{-3}$	0.065
4	$3.167 \cdot 10^{-5}$	$1.582 \cdot 10^{-3}$
5	$2.867 \cdot 10^{-7}$	$1.433 \cdot 10^{-5}$
6	$9.866 \cdot 10^{-10}$	$4.933 \cdot 10^{-8}$

5 Graphical results $\beta1 := 0, 0.1 .. 6$ $\beta nt := 3.8$ $pnt := p1(\beta nt)$ $pnt = 7.235 \times 10^{-5}$





CHAPTER II - ELEMENTARY METHODS OF STRUCTURAL RELIABILITY I

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Summary

Elementary methods of structural reliability are described considering a fundamental case of two random variables when the limit state function is formulated as a difference between the resulting structural resistance and load effect. The initial assumption of normal distribution of both resulting variables is generalised to any type of probability distribution. The described computational procedures are illustrated by a number of numerical examples, which are supplemented by MATHCAD and EXCEL sheets. An extension of the elementary methods of structural reliability is presented in Annex B.

1 INTRODUCTION

1.1 Background materials

Fundamental concepts and procedures of structural reliability are well described in a number of national standards, in the new European document EN 1990 [1] and International Standard ISO 2394 [2]. Additional information may be found in the background document developed by JCSS [3] and in recently published handbook to EN 1990 [4]. Guidance on application of the probabilistic methods of structural reliability may be found in publications and working materials developed by JCSS [5] and in relevant literature listed in [4] and [5].

1.2 General principles

The theory of structural reliability considers all basic variables as random quantities having appropriate types of probability distribution. Different types of distributions should be considered for actions, material properties and geometrical data. In addition, model uncertainties of actions and resistance models should be taken into account. Prior theoretical models of basic variables and procedures for probabilistic analysis are indicated in JCSS documents [5].

2 FUNDAMENTAL CASES OF STRUCTURAL RELIBILITY

2.1 General

The fundamental task of the theory of structural reliability concerns a basic requirement for the relation between the action effect E and the structural resistance R written in the form of inequality

$$E < R \tag{1}$$

Condition (1) describes a desirable (satisfactory, safe) state of a considered structural component. It is assumed that structural failure occurs when the condition (1) is not satisfied. Thus, an assumed sharp (unambiguous) distinction between a desirable (safe) and undesirable (failure) state of the structure is given as

$$R - E = 0 \tag{2}$$

Equation (2) represents a fundamental form of the failure boundary called the limit state (performance) function (see also Chapter I of this Handbook 2). It should be noted that for some structural members and materials the assumption of sharp failure boundary might be rather artificial and can be accepted as an approximation only. Such a case is indicated in the following Example 1.

Example 1.

A steel rod indicated in Figure 1 has a tensile resistance $R = \pi\, d^2 f_y / 4$, where d denotes the diameter of the rod and f_y the yield point. The rod is loaded by a weight $E = V\rho$, where V denotes the volume and ρ the bulk weight density of the load. Thus the inequality (1) has the form

$$V\rho < \pi\, d^2 f_y / 4$$

The limit state function (2) can be then written as

$$\pi\, d^2 f_y / 4 - V\rho = 0$$

In this example, the limit state is defined as the state when the stress in the rod reaches the yield point f_y. This simplification is accepted in many common cases, but (depending on a type of structural steel) it may not correspond to the actual failure of the rod. In particular when structural steel with significant ductility and strain hardening is used, then a failure (rupture) will occur when the stress reaches the ultimate strength of the steel, which is a considerably greater than the yield point.

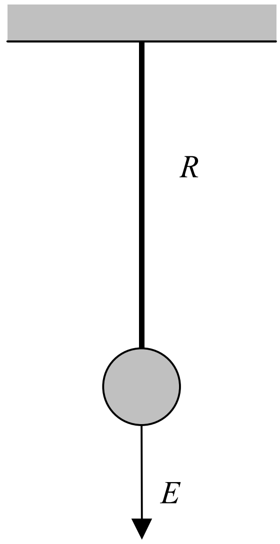

Figure 1. A rod.

Attached MATHCAD sheets SteelRod.mcd, DesVRod.mcd may be used to make all numerical calculations.

Both the variables E and R are generally random variables and the validity of inequality (1) cannot be guaranteed absolutely, i.e. with the probability equal to 1 (the total certainty). Therefore, it is necessary to accept the fact that the limit state described by equation (2) may be exceeded and failure may occur with a certain small probability. The essential objective of the reliability theory is to assess the probability of failure P_f and to find the necessary conditions for its limited magnitude. For the simple condition in the form of inequality (1), the probability of failure may be formally written as

$$P_f = \mathrm{P}(E > R) \tag{3}$$

The random character of the action effect E and the resistance R, both expressed in terms of a suitable variable (performance indicator) X (i.e. stress, force, bending moment, deflection) is usually described by appropriate distribution function, i.e. by distribution functions $\Phi_E(x)$, $\Phi_R(x)$ and by corresponding probability density functions $\varphi_E(x)$, $\varphi_R(x)$, where x denotes a general point of the considered variable X used to express both the variables E and R. Distributions of variables E and R further depend on appropriate parameters, e.g. on moment parameters μ_E, σ_E, ω_E, μ_R, σ_R and ω_R. Let us further assume that E and R are mutually independent (which may be provided by appropriate transformation).

Figure 2 shows an example of the probability density functions of both the variables E and R and their mutual location. Types of distribution and their parameters shown in Figure 2 are just indicative information. In particular, the moment parameters (the means and standard deviations) may be considered as relative values given as a percentage of the resistance mean μ_R (i.e. normalised by μ_R).

Note, that the probability density functions $\varphi_E(x)$ and $\varphi_R(x)$ shown in Figure 2 overlap each other and, therefore, it is clear that unfavourable realizations of variables E and R, denoted by small letters e and r, may occur in such a way that $e > r$, i.e. the load effect is greater than the resistance and failure will occur. Obviously in order to keep the failure probability $P_f = P(E>R)$ within an acceptable limits, the parameters of variables E and R must satisfy certain conditions (concerning the mutual position and variances of both distributions) depending on the types of distribution.

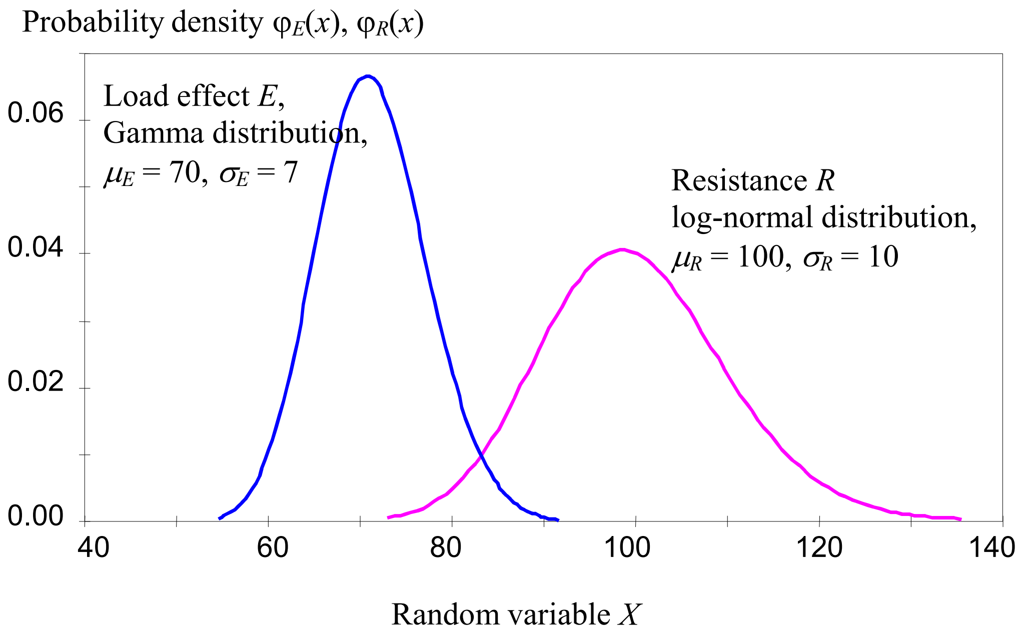

Figure 2. Action effect E and resistance R as random variables.

The desired conditions will certainly include the trivial inequality $\mu_E < \mu_R$ (see Figure 2). Obviously, this "requirement for mutual position" of both distributions is not sufficient to ensure specified failure probability P_f. The correct conditions should certainly include also conditions for variances of both variables. This will be clarified by the following discussion of fundamental cases of structural reliability.

2.2 Fundamental cases of one random variable

First, consider a special case when one of the variables E and R, say the action effect E, has a very low (negligible) variability comparing to the variability of resistance R. Then E may be considered as non-random (deterministic) variable, i.e. such a variable that attains a certain fixed value E_0 ($E = E_0$) in its every realization. This assumption may certainly be considered as an approximation of some practical cases. One of these cases is the loaded steel rod from Example 1, where the weight of the suspended mass can be determined with sufficient accuracy (i.e. without any significant uncertainty). This special case is illustrated in Figure 3, where the action effect is indicated by a fixed value e_0 = 80 (μ_E = 80, σ_E = 0) and the resistance by the lognormal distribution having the mean μ_R = 100, σ_R = 10 (all numerical data being normalised to dimensionless quantities).

The probability of failure P_f for the special case of deterministic load effect of actions shown in Figure 3 may be assessed directly from the distribution function $\Phi_R(x)$ similarly as in the case of a fractile. The value e_0 may be simply considered as the fractile of the resistance R for which the probability P_f may be calculated using equation

$$P_f = P(R < e_0) = \Phi_R(e_0) \tag{4}$$

The value of distribution function $\Phi_R(E_0)$ is usually assessed from tables for a standardized random variable U, for which the value u_0 corresponding to E_0 is computed. It follows from the general transformation formula

$$u_0 = (e_0 - \sigma_R) / \sigma_R \tag{5}$$

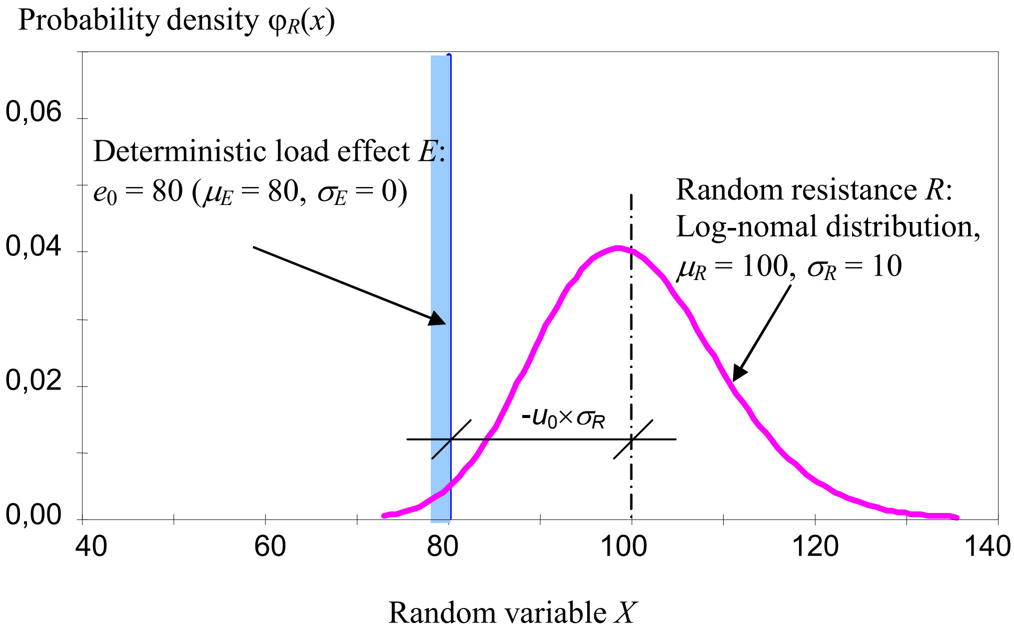

Figure 3. Deterministic effect of actions E and random resistance R.

The probability of failure is then given as

$$P_f = P(R < e_0) = \Phi_R(e_0) = \Phi_U(u_0) \tag{6}$$

where $\Phi_U(u_0)$ is the value of distribution function of a standardized random variable of the appropriate distribution (e.g. normal or log-normal).

Note that the value $-u_0$ is the distance of the fixed value E_0 of action effect E from the mean μ_R of resistance R expressed in the units of standard deviation σ_R. If the distribution of resistance R is normal, then the defined distance is called the reliability index β

$$\beta = (\mu_R - e_0) / \sigma_R \tag{7}$$

and the probability of failure may be expressed by the relation

$$P_{\mathrm{f}} = \mathrm{P}(R < e_0) = \Phi_U(-\beta) \tag{8}$$

In general the reliability index β is defined as the negative value of a standardized normal variable corresponding to the probability of failure P_{f}. Thus, the following relationship is accepted as a definition (see Chapter I in this Handbok)

$$\beta = -\Phi_U^{-1}(p_f) \tag{9}$$

where $-\Phi_u^{-1}(p_{\mathrm{f}})$ denotes the inverse standardised normal distribution function. At present the reliability index β defined by equation (9) is a commonly used measure of structural reliability in several national and international documents (see also previous Chapter I of this Handbook 2). Note, however, that the probability distribution of the resistance R may differ from the normal distribution.

Example 2.

Consider that resistance R has the mean μ_R = 100 (expressed in dimensionless units), standard deviation σ_R =10 (the coefficient of variation is V_R = 0,10). For the deterministic action effect it holds that e_0 = 80 (see Figure 3). If R has normal distribution, then the reliability index follows directly from equation (7)

$$\beta = (100 - 80) / 10 = 2$$

and probability of failure follows from relation (8)

$$P_{\mathrm{f}} = \mathrm{P}(R < 80) = \Phi_u(-2) = 0{,}023$$

where $\Phi_u(-2)$ is the value of the distribution function of the standardized normal distribution for $u = -2$. However, if the distribution of R is not normal but lognormal with the lower limit at zero (skewness $\omega_R = 3V_R + V_R^{\ 3}$= 0,301 [9]), then it follows from equation (5)

$$u_0 = (80 - 100) / 10 = -2$$

The probability of failure P_{f} is then given as

$$P_{\mathrm{f}} = \mathrm{P}(R < 80) = \Phi_{\mathrm{LN,U}}(-2) = 0{,}014$$

where $\Phi_{\mathrm{LN},U}(-2)$ is the distribution function of the standardized random variable U with log-normal distribution having the lower bound at zero (the skewness ω = 0,301). The resulting probabilities do not much differ but their values are rather high.

If the fixed value of the action effect decreases to e_0 = 70, then for normal distribution of resistance R the reliability index is β = 3 and probability of failure is

$$P_{\mathrm{f}} = \mathrm{P}(R < 70) = \Phi_{\mathrm{u}}(-3) = 0{,}00135$$

If the distribution of resistance R is log-normal with the lower limit at zero, then

$$P_{\mathrm{f}} = \mathrm{P}(R < 70) = \Phi_{\mathrm{LN},U}(-3) = 0{,}00021$$

The reliability index defined by equation (9) is then $\beta = -\Phi_U^{-1}(0{,}00021) = 3{,}53$, i.e. greater than the value 3, which holds if normal distribution of resistance R is assumed.

Obviously, when the load effect is only e_0 = 70 the resulting failure probabilities are remarkably lower than in the case when e_0 = 80. Furthermore, the numerical example also shows that the assumption concerning the type of distribution plays an important role and may be, in some cases, decisive.

2.3 Fundamental case of two random variables

Assume that both basic variables, the action effect E and the resistance R are random variables. Then it is generally more complicated to assess the probability of failure defined by equation (3). A simple solution can be obtained assuming a normal distribution for both E and R. Then also the difference

$$Z = R - E \tag{10}$$

called the safety margin, has the normal distribution with parameters

$$\mu_Z = \mu_R - \mu_E \tag{11}$$

$$\sigma_Z^2 = \sigma_R^2 + \sigma_E^2 + 2\rho_{RE}\sigma_R^2\sigma_E^2 \tag{12}$$

where ρ_{RE} is the coefficient of correlation of R and E. It is often assumed that R and E are mutually independent and $\rho_{RE} = 0$. Equation (3) for the probability of failure P_{f} can now be modified to

$$P_{\mathrm{f}} = \mathrm{P}(E > R) = \mathrm{P}(Z < 0) = \Phi_Z(0) \tag{13}$$

and the whole problem is reduced to determining the distribution function $\Phi_Z(z)$ for $z = 0$, which leads to the probabilities of the safety margin Z being negative. The distribution function $\Phi_Z(0)$ is usually determined by transformation of the variable Z to standardised random variable U. Using this equation, the value u_0 corresponding to the value $g = 0$ is

$$u_0 = (0 - \mu_Z) / \sigma_Z = - \mu_Z / \sigma_Z \tag{14}$$

The probability of failure is then given as

$$P_{\mathrm{f}} = \mathrm{P}(R < E) = \Phi_Z(0) = \Phi_{\mathrm{U}}(\mathrm{u}_0) \tag{15}$$

The probability density function $\varphi_Z(z)$ of the safety margin Z is shown in Figure 4, where the grey area under the curve $\varphi_Z(z)$ corresponds to the failure probability P_{f}.

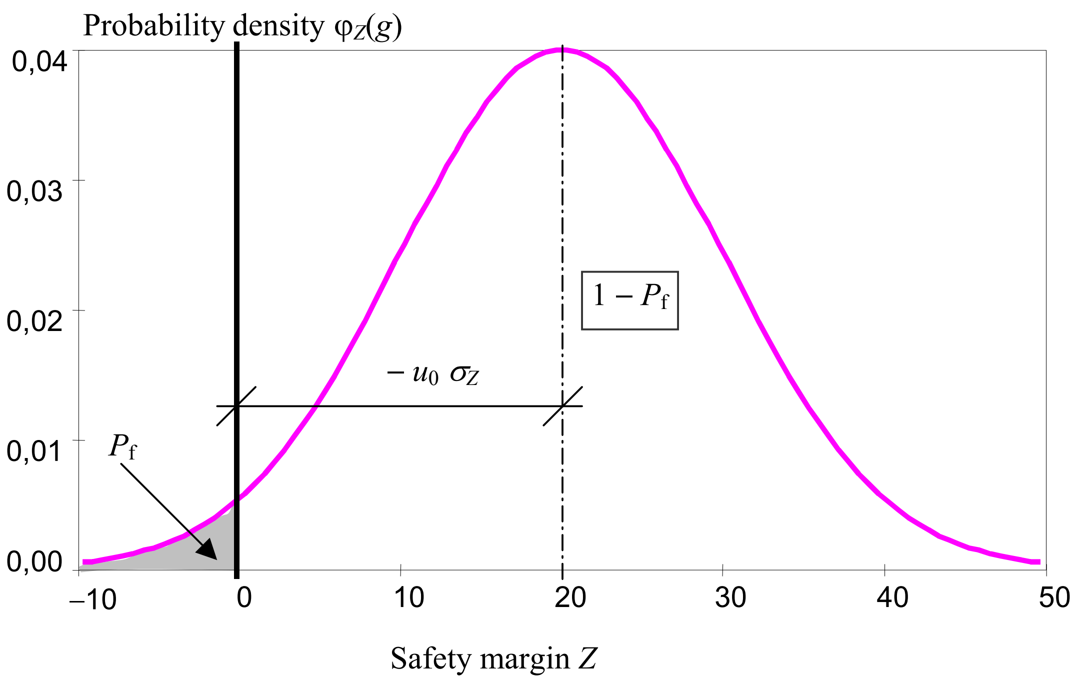

Figure 4. Distribution of the safety margin Z.

Assuming that Z has a normal distribution, the value $-u_0$ is called the reliability index, which is commonly denoted by the symbol β. In case of a normal distribution of the safety margin Z, it follows from equations (11), (12) and (14) that the reliability index β is given by a simple relationship

$$\beta = \mu_Z / \sigma_Z = \frac{\mu_R - \mu_E}{\sqrt{\sigma_R^2 + \sigma_E^2 + 2\rho_{RE}\sigma_R^2\sigma_E^2}} \tag{16}$$

If the quantities R and E are mutually independent, then the coefficient of correlation ρ_{RE} vanishes (ρ_{RE}=0). Thus, the reliability index β is the distance of the mean μ_Z of the safety margin Z to the origin, given in the units of the standard deviation σ_Z.

Example 3.

Consider again the Example 2, in which the resistance R and the load effect E are mutually independent random variables (ρ_{RE}=0) having normal distribution. The resistance R has the mean μ_R = 100, variance σ_R = 10 (coefficient of variation is therefore only w = 0,10), and the effect of actions E has the mean μ_E = 80 and σ_E = 8 (all expressed in dimensionless units). It follows from equation (11) and (12) that

$$\mu_Z = 100 - 80 = 20$$

$$\sigma_Z^2 = 10^2 + 8^2 = 12{,}81^2$$

As both the basic variables R and E have normal distributions, the reliability index β follows directly from equation (16)

$$\beta = 20 / 12{,}81 = 1{,}56$$

and the probability of failure follows from relation (8)

$$P_\mathrm{f} = \mathrm{P}(Z < 0) = \Phi_U(-1{,}56) = 0{,}059$$

If the variables E and R are not normal, then the distribution of the safety margin G is not normal either and then the above-described procedure has to be modified. In a general case, numerical integration or transformation of both variables into variables with normal distribution can be used. The transformation into a normal distribution is primarily used in software products.

An approximate simple procedure can be used for a first assessment of the failure probability P_f. The safety margin Z may be approximated by a three-parameter lognormal distribution. Assume that the distributions of E and R depend on the moment parameters μ_E, σ_E, ω_E, μ_R, σ_R and ω_R. The mean and variance of the safety margin Z may be assessed from the previous equations (11) and (12), which hold for variables with an arbitrary distribution. Assuming mutual independence of E and R, the skewness ω_Z of the safety Z may be estimated using the approximate formula (see Annex A - Basic statistical concepts and techniques in this Handbook 2)

$$\omega_Z = \frac{\sigma_R^3\omega_R - \sigma_E^3\omega_E}{(\sigma_R^2 + \sigma_R^2)^{3/2}} \tag{17}$$

Then it is assumed that the safety margin Z can be described with sufficient accuracy by a log-normal distribution with determined moment parameters μ_Z, σ_Z and ω_Z (equations (11), (12) and (17)). It shows that this approximation offers satisfactory results if the probability of failure is not too small.

Example 4.

Consider a tie rod having a resistance R under a suspended load of weight E. Let R be a log-normal variable with origin at zero having the parameters (expressed again in relative dimensionless units) μ_R = 100 and σ_R = 10 (and therefore ω_R = 0,301), E has Gumbel distribution with moment parameters μ_E = 50 and σ_E = 10 (for Gumbel distribution [9] has the positive skewness ω_E = 1,14).

The moment parameters of the safety margin are assessed according to equations (11), (12) and (17)

$$\mu_Z = \mu_R - \mu_E = 100 - 50 = 50$$

$$\sigma_G^2 = \sigma_R^2 + \sigma_E^2 = 10^2 + 10^2 = 14{,}14^2$$

$$\omega_Z = \frac{\sigma_R^3 \omega_R - \sigma_E^3 \omega_E}{\left(\sigma_R^2 + \sigma_E^2\right)^{3/2}} = \frac{10^3 \times 0{,}301 - 10^3 \times 1{,}14}{\left(10^2 + 10^2\right)^{3/2}} = -0{,}30$$

For a standardized random variable it follows from equation (14) that

$$u_0 = -\mu_Z / \sigma_Z = -50 / 14{,}14 = -3{,}54$$

For a log-normal distribution having the skewness μ_Z = - 0,30 it holds that

$$P_{\mathrm{f}} = \mathrm{P}(R < E) = \Phi_{\mathrm{LN},\underline{\mathrm{U}}}(-3{,}54) = 0{,}00101$$

which corresponds to the reliability index β = 3,09. A more precise result obtained by application of the software VaP [7] is P_{f} = 0,00189.

However, when skewness is not taken into account in the assessment of failure probability and the normal distribution is assumed, it follows that

$$P_{\mathrm{f}} = \mathrm{P}(R < E) = \Phi_{\mathrm{U}}(-3{,}54) = 0{,}00020$$

which differs significantly from the result when the log-normal distribution was assumed.

Attached MATHCAD sheets StRod.mcd, DesVRod.mcd may be used to make all numerical calculations.

3 EXACT SOLUTION FOR TWO RANDOM VARIABLES

In the case of two random variables E and R having any distribution, the exact determination of the failure probability P_{f}, defined by equation (3), may be obtained by probability integration. Figure 5 is used to explain this procedure. Let the event A denote the occurrence of the action effect E in the differential interval $<x, x+\mathrm{d}x>$. Probability of the event A is given as

$$\mathrm{P}(A) = \mathrm{P}(x < E < x+\mathrm{d}x) = \varphi_E(x)\,\mathrm{d}x \tag{18}$$

Let us denote B as the event that resistance R occurs within the interval $<-\infty, x>$. Probability of the event B is [9] given as

$$\mathrm{P}(B) = \mathrm{P}(R < x) = \Phi_R(x) \tag{19}$$

The differential increment of failure probability $\mathrm{d}P_{\mathrm{f}}$ corresponding to the occurrence of the variable E in the interval $<x, x+\mathrm{d}x>$ is given by the probability of the simultaneous occurrence of the events A and B, i.e. by the probability of their intersection $A \cap B$. According to the principle of multiplication of probabilities [10], it holds that

$$\mathrm{d}P_{\mathrm{f}} = \mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B) = \mathrm{P}(x < E < x+\mathrm{d}x)\,\mathrm{P}(R < x) = \Phi_R(x)\,\varphi_E(x)\,\mathrm{d}x \tag{20}$$

The above-mentioned assumption of mutual independence of the variables *E* and *R*, and thus also of the events *A* and *B*, is applied here.

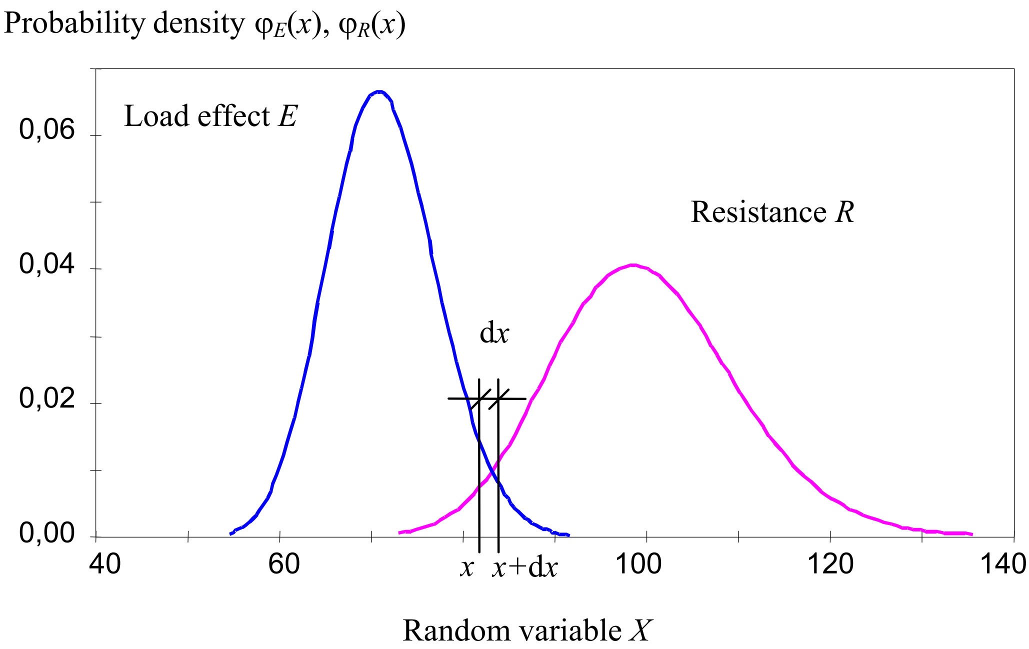

Figure 5. Distribution of variables *E* and *R*.

The integration of the differential relationship (20) over the interval in which both variables *E* and *R* occur simultaneously (generally the interval <-∞, ∞>) leads to the relation

$$p_{\mathrm{f}} = \int_{-\infty}^{\infty} \Phi_R(x)\varphi_E(x)\mathrm{d}x \tag{21}$$

the integration of the relation (21) usually has to be carried out numerically or using the simulation methods (e.g. direct Monte Carlo methods).

Attached MATHCAD sheet PrLnLn.mcd offers a simple programme that can be used to evaluate the numerical integration of relation (21) assuming that both the variables *E* and *R* can be described (at least approximately) by the general (three-parameter) lognormal distribution.

Example 5.

The action effect *E* and the resistance *R* are described by a log-normal distribution with the same parameters as in Example 4 (the Gumbel distribution for *E* was simply substituted by the log-normal distribution having the same parameters). The approximate solution in Example 4, based on log-normal distribution with the lower bound at zero, leads to the probability of failure $P_{\mathrm{f}} = \mathrm{P}(R < E) = \Phi_{\mathrm{LN},U}(-3{,}54) = 0{,}00101$. The numerical integration according to relation (21) using the programme MATHCAD leads to a solution $P_{\mathrm{f}} = \mathrm{P}(R < E) = 0{,}000792$, the programme VaP suggests a solution $P_{\mathrm{f}} = \mathrm{P}(R < E) = 0{,}000707$, which can be considered as a very good approximation.

The probability of failure P_{f} assessed by the direct integration may be determined using MATHCAD sheet PrLnLn.mcd for the given parameters of variables *E* and *R* (μ_R =

100, σ_R = 10, μ_E = 50 and σ_E = 10). Variation of probability of failure P_f with the coefficients of skewness ω_E and ω_R is shown in Figure 6.

It follows from Figure 6 that the probability of failure P_f depends greatly on the skewnesses ω_E and ω_R (therefore on assumed theoretical models), and in practical conditions, can differ by several orders of magnitude, even when the means and standard deviations of the variables E and R remain the same.

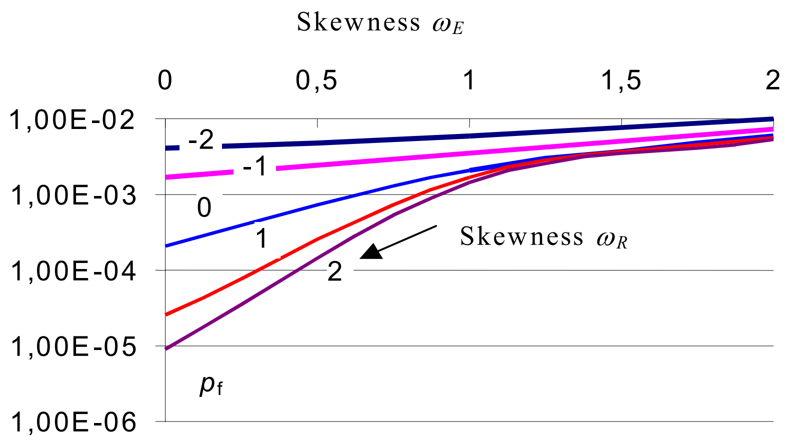

Figure 6. Probability of failure P_f versus the coefficients of skewness ω_E and ω_R for μ_R = 100, μ_R = 10, σ_E = 50 and σ_E = 10.

It appears that the determination of the failure probability in the case of a simple example described by inequality (1), where only two random variables E and R are involved, is easy only when both variables are normally distributed. If they have other distributions, the exact solution is more complicated and the resulting values depend significantly on the assumed types of distributions. The approximate solution assuming for E and R a general (three parameter) lognormal distribution provides a good first estimate of the failure probability. The obtained values should be, however, verified by more exact procedures considering appropriate theoretical models of E and R.

4 CONCLUDING REMARKS

Elementary methods of the structural reliability can be used to assess the reliability of fundamental cases of two random variables when the limit state function is formulated as the difference between the resulting structural resistance and load effect.

Basic principles of the reliability theory provide operational techniques that can be used for estimating the partial factors of basic variables. The assessment of various reliability measures in the new structural design codes is, however, partly based on historical and empirical experiences. Obviously the past experience depends on local conditions including climatic actions and traditionally used construction materials and, consequently, might be in different countries considerably diverse. That is why number of reliability elements and parameters in the present suite of European standards are open for national choice.

REFERENCES

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ATTACHMENTS

1. MATHCAD sheet "SteelRod.mcd"

Mathcad sheet SteelRod is intended to investigate an effect of the partial factor γ_G on reliability of a steel rod exposed to permanent load G.

2. MATHCAD sheet "DesVRod.mcd"

Mathcad sheet DesVRod is intended to investigate of sensitivity factor α_E and α_R and design values E_d and R_d.

3. MATHCAD sheet "PrLnLn.mcd"

Mathcad sheet PrLnLn is intended for calculation of the failure probability Pf = P$\{E>R\}$based on approximation of E and R by three parameter lognormal distribution.

Attachment 1 - MATHCAD sheet "SteelRod.mcd"

A steel rod under a permanent load G - parameter study of γ_G

1 Design of a rod cross section area $A = G_d / f_d$

Design input data: $Gk := 1$ $\quad \gamma G := 1.0, 1.05 .. 1.6$ (parameter) $\quad fk := 235$ $\quad \gamma m := 1.10$ $\quad fd := \dfrac{fk}{\gamma m}$

Design of the cross section area $\quad A(\gamma G) := \dfrac{(Gk \cdot \gamma G)}{fd}$ $\quad$ Check: $\boxed{A(1.35) = 6.32 \times 10^{-3}}$

2 Parameters of basic varibles G and f

Parameters of G and f: $\mu G := Gk$ $\quad vG := 0.1$ $\quad \sigma G := vG \cdot \mu G$ $\quad \omega := \dfrac{280}{235}$ $\quad \mu f := \omega \cdot fk$ $\quad vf := 0.08$ $\quad \sigma f := vf \cdot \mu f$

Model uncertainty: $\mu XS := 1$ $\quad \sigma XS := 0$ $\quad \mu XR := 1$ $\quad \sigma XR := 0.00$ $\quad vXR := \dfrac{\sigma XR}{\mu XR}$ $\quad vXS := \dfrac{\sigma XS}{\mu XS}$

3 Parameters of the resistance R and load effect E

The mean of R and E $\quad \mu R(\gamma G) := \mu f \cdot \mu XR \cdot A(\gamma G)$ $\quad \mu E := \mu G \cdot \mu XS$ $\quad \boxed{\mu R(1.35) = 1.77}$ $\boxed{\mu E = 1}$

CoV: $vR := \sqrt{vXR^2 + vXR^2 \cdot vf^2 + vf^2}$ $\quad vE := \sqrt{vXS^2 + vXS^2 \cdot vG^2 + vG^2}$ $\quad$ Check: $\boxed{vR = 0.08}$ $\boxed{vE = 0.1}$

Skewness of R for lognormal and E for gamma distribution: $\quad \alpha R := 3 \cdot vR + vR^3$ $\quad \alpha E := 2 \cdot vE$

4 Parameters of the reliability margin $g = R - E$

$\mu g(\gamma G) := \mu R(\gamma G) - \mu E$ $\quad \sigma R(\gamma G) := vR \cdot \mu R(\gamma G)$ $\quad \sigma E := vE \cdot \mu E$ $\quad \boxed{\sigma R(1.35) = 0.14}$

$\sigma g(\gamma G) := \sqrt{(\sigma R(\gamma G))^2 + (\sigma E)^2}$ $\quad \boxed{\mu g(1.35) = 0.77}$ $\boxed{\sigma g(1.35) = 0.17}$

$$\alpha g(\gamma G) := \frac{\alpha R \cdot \sigma R(\gamma G)^3 - \alpha E \cdot \sigma E^3}{\sigma g(\gamma G)^3} \qquad \boxed{\alpha g(1.35) = 0.09}$$

5 Reliability assessment without integration

Reliability index assuming normal distribution of g (a first estimate)

$\beta 0(\gamma G) := \dfrac{\mu g(\gamma G)}{\sigma g(\gamma G)}$ $\quad Pf0(\gamma G) := \mathrm{pnorm}(-\beta 0(\gamma G), 0, 1)$ $\quad$ Check: $\boxed{\beta 0(1.35) = 4.44}$

Reliability index assuming three parameter lognormal distribution of g (a refine estimate)

Parameter C of three parametr lognormal distributon of g:

$$C(\gamma G) := \frac{\left(\sqrt{\alpha g(\gamma G)^2 + 4} + \alpha g(\gamma G)\right)^{\frac{1}{3}} - \left(\sqrt{\alpha g(\gamma G)^2 + 4} - \alpha g(\gamma G)\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}}$$

Parameters of transformed variable: $\quad mg(\gamma G) := -\ln(|C(\gamma G)|) + \ln(\sigma g(\gamma G)) - (0.5) \cdot \ln\left(1 + C(\gamma G)^2\right)$

$sg(\gamma G) := \sqrt{\ln\left(1 + C(\gamma G)^2\right)}$ $\quad x0(\gamma G) := \mu g(\gamma G) - \dfrac{1}{C(\gamma G)} \sigma g(\gamma G)$ $\quad$ Check: $\boxed{x0(1.35) = -4.85}$

$Pf1(\gamma G) := \mathrm{plnorm}(0 - x0(\gamma G), mg(\gamma G), sg(\gamma G))$ $\quad \beta 1(\gamma G) := -\mathrm{qnorm}(Pf1(\gamma G), 0, 1)$ $\quad \boxed{\beta 1(1.35) = 4.76}$

6 Reliability assessment using integration

Assuming normal distribution for *E*: $En(x) := dnorm(x, \mu E, \sigma E)$

Assuming gamma distribution for *E*: $k := \left(\frac{\mu E}{\sigma E}\right)^2$ $\lambda := \left(\frac{\mu E}{\sigma E^2}\right)$ $Eg(x) := dgamma(\lambda \cdot x, k) \cdot \lambda$

Assuming lognormal distribution of R having the lower limit at a (0 default):

$a(\gamma G) := \mu R(\gamma G) \cdot 0.0$ $C(\gamma G) := \frac{\sigma R(\gamma G)}{(\mu R(\gamma G) - a(\gamma G))}$ $aR(\gamma G) := C(\gamma G)^3 + 3 \cdot C(\gamma G)$

$m(\gamma G) := \ln(\sigma R(\gamma G)) - \ln(C(\gamma G)) - (0.5) \cdot \ln\left(1 + C(\gamma G)^2\right)$ $s(\gamma G) := \sqrt{\ln\left(1 + C(\gamma G)^2\right)}$

Probability lognormal distribution of *R* $Rln(x, \gamma G) := plnorm[(x - a(\gamma G)), m(\gamma G), s(\gamma G)]$

Failure probability Prob{*R*<*E*} and reliability index β $\beta t := 3.8$

E has normal, *R* lognormal distribution

$$Pfn(\gamma G) := \int_0^{\infty} En(x)\, Rln(x, \gamma G)\, dx \qquad \beta n(\gamma G) := -qnorm(Pfn(\gamma G), 0, 1)$$

E has gamma, *R* lognormal distribution

$$Pfg(\gamma G) := \int_0^{\infty} Eg(x)\, Rln(x, \gamma G)\, dx \qquad \beta g(\gamma G) := -qnorm(Pfg(\gamma G), 0, 1)$$

7 Parametric study of γG

Check:

$\boxed{Pf0(1.35) = 4.51 \times 10^{-6}}$

$\boxed{Pf1(1.35) = 9.68 \times 10^{-7}}$

$\boxed{Pfg(1.35) = 1.98 \times 10^{-6}}$

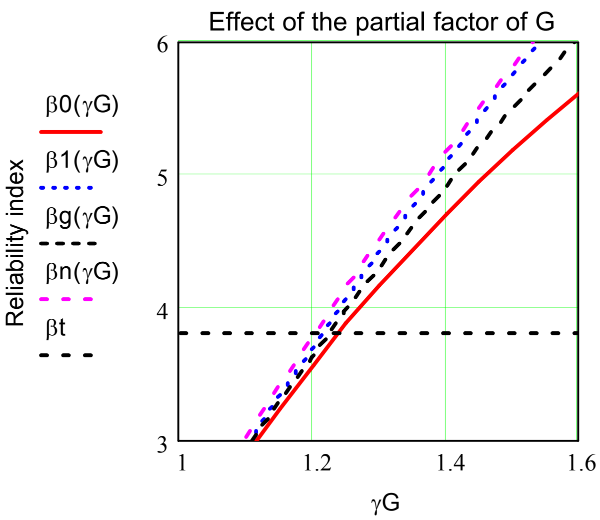
$\boxed{Pfn(1.35) = 6.24 \times 10^{-7}}$

$\boxed{\beta 0(1.35) = 4.44}$

$\boxed{\beta 1(1.35) = 4.76}$

$\boxed{\beta g(1.35) = 4.61}$

$\boxed{\beta n(1.35) = 4.85}$



Note: Reliability assessment assuming normal distribution for *E* and *R* seems to be on a safe side (leads to a lower bound for β], while assessment assuming three parameter distribution for the reliability margin *g* seems to provide a more realistic estimate.

Attachment 2 - MATHCAD sheet "DesVRod.mcd"

A steel rod under a permanent load *G* - sensitivity factors α_E and α_R

1 Design of a hanger cross section area $A = G_d / f_d$

Design input data: $Gk := 1$ $\quad \gamma G := 1.0, 1.05..\ 1.6$ (parameter) $\quad fk := 235$ $\quad \gamma m := 1.10$ $\quad fd := \dfrac{fk}{\gamma m}$

Design of the cross section area $\quad A(\gamma G) := \dfrac{(Gk \cdot \gamma G)}{fd}$ $\quad$ Check: $\boxed{A(1.35) = 6.32 \times 10^{-3}}$

2 Parameters of basic variables *G* and *f*

Parameters of *G* and *f*: $\mu G := Gk$ $\quad vG := 0.1$ $\quad \sigma G := vG \cdot \mu G$ $\quad \omega := \dfrac{280}{235}$ $\quad \mu f := \omega \cdot fk$ $\quad vf := 0.08$ $\quad \sigma f := vf \cdot \mu f$

Model uncertainty: $\mu XS := 1$ $\quad \sigma XS := 0$ $\quad \mu XR := 1$ $\quad \sigma XR := 0.00$ $\quad vXR := \dfrac{\sigma XR}{\mu XR}$ $\quad vXS := \dfrac{\sigma XS}{\mu XS}$

3 Parameters of the resistance *R* and load effect E

The mean of *R* and *E* $\quad \mu R(\gamma G) := \mu f \cdot \mu XR \cdot A(\gamma G)$ $\quad \mu E := \mu G \cdot \mu XS$ $\quad \boxed{\mu R(1.35) = 1.77}$ $\boxed{\mu E = 1}$

CoV: $vR := \sqrt{vXR^2 + vXR^2 \cdot vf^2 + vf^2}$ $\quad vE := \sqrt{vXS^2 + vXS^2 \cdot vG^2 + vG^2}$ $\quad$ Check: $\boxed{vR = 0.08}$ $\boxed{vE = 0.1}$

Skewness of *R* for lognormal and *E* for gamma distribution: $\quad \alpha R := 3 \cdot vR + vR^3$ $\quad \alpha E := 2 \cdot vE$

4 Parameters of the reliability marging = *R* - *E*

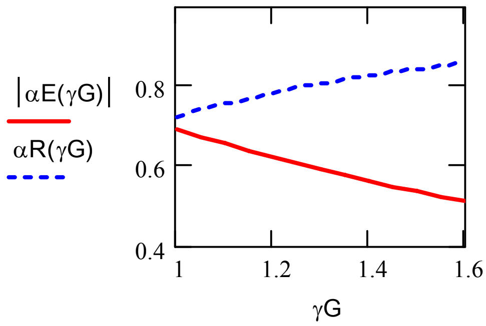
$\mu g(\gamma G) := \mu R(\gamma G) - \mu E$ $\quad \sigma R(\gamma G) := vR \cdot \mu R(\gamma G)$ $\quad \sigma E := vE \cdot \mu E$ $\quad \boxed{\sigma R(1.35) = 0.14}$

$\sigma g(\gamma G) := \sqrt{(\sigma R(\gamma G))^2 + (\sigma E)^2}$ $\quad \boxed{\mu g(1.35) = 0.77}$ $\boxed{\sigma g(1.35) = 0.17}$

$$\alpha g(\gamma G) := \frac{\alpha R \cdot \sigma R(\gamma G)^3 - \alpha E \cdot \sigma E^3}{\sigma g(\gamma G)^3}$$ $\quad \boxed{\alpha g(1.35) = 0.09}$

5 Sensitivity coefficients α_E and α_R

$$\alpha E(\gamma G) := \frac{-\sigma E}{\sigma g(\gamma G)} \qquad \alpha R(\gamma G) := \frac{\sigma R(\gamma G)}{\sigma g(\gamma G)}$$



Note that the sensitivity factor α_E is shown with the oposite sign (as a positive quantity).

6 Design values E_d and R_d

EC 1990 recommendation: $\beta := 3.8$ $\alpha E0 := -0.7$ $\alpha R0 := 0.8$

$$Ed(\gamma G) := \mu E - \alpha E(\gamma G)\,\beta \cdot \sigma E \qquad Rd(\gamma G) := \mu R(\gamma G) - \alpha R(\gamma G)\,\beta \cdot \sigma R(\gamma G)$$

$$Ed0(\gamma G) := \mu E - \alpha E0\beta \cdot \sigma E \qquad Rd0(\gamma G) := \mu R(\gamma G) - \alpha R0\beta \cdot \sigma R(\gamma G)$$

$$Rd0ln(\gamma G) := \mu R(\gamma G) \cdot \exp(-\alpha R0\,\beta \cdot vR)$$

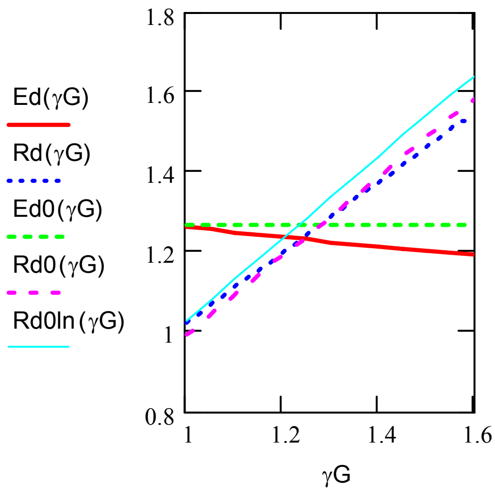
Check: $Ed(1.35) = 1.22$

$Ed0(1.35) = 1.27$

$Rd(1.35) = 1.33$

$Rd0(1.35) = 1.34$

$Rd0ln(1.35) = 1.39$



Notes: 1) Figure shows that the partial factor γ_G should be greater than about 1,25 otherwise the design value of the load effect E_d would be greater than the design value of the resistance Rd.
2) The design value of the resistance R_d determined assuming lognormal distribution with the lower bound at zero is greater than R_d determined assuming the normal distribution.

Attachment 3 - MATHCAD sheet "PrLnLn.mcd"

Failure probability pf=P{E>R} for log-normal distribution L**N(μ,σ,α)** of E and R

1. Input parameters for E and R : $\mu E := 50 \quad \sigma E := 10. \quad \alpha E := 0, 0.1..2 \quad x := 0, 0.1\mu E..3\mu E \quad wE := \frac{\sigma E}{\mu E}$

$$\mu R := 100 \quad \sigma R := 10. \quad \alpha R := -1, -0.9..2 \quad x := 0, 0.1\mu R..3\mu R \quad wR := \frac{\sigma R}{\mu R}$$

Distribution parameter C given by the skewness αE:

$$CE(\alpha E) := \frac{\sqrt[3]{\left(\sqrt{\alpha E^2 + 4} + \alpha E\right)} - \sqrt[3]{\sqrt{\alpha E^2 + 4} - \alpha E}}{\sqrt[3]{2}}$$

Check: $CE(0.301) = \blacksquare$

Distribution bound x0E:

$$x0E(\alpha E) := \left(\begin{array}{|l} \mu E - \frac{\sigma E}{CE(\alpha E)} \text{ if } \alpha E \neq 0 \\ \mu E - 6 \cdot \sigma E \text{ otherwise} \end{array} \right)$$

$x0E(0) = \blacksquare$

Distribution parameter CR given by the skewness αR:

$$CR(\alpha R) := \frac{\sqrt[3]{\left(\sqrt{\alpha R^2 + 4} + \alpha R\right)} - \sqrt[3]{\sqrt{\alpha R^2 + 4} - \alpha R}}{\sqrt[3]{2}}$$

Check: $CR(0.3) = \blacksquare$

Distribution bound x0R:

$$x0R(\alpha R) := \left(\begin{array}{|l} \mu R - \frac{\sigma R}{CR(\alpha R)} \text{ if } \alpha R \neq 0 \\ \mu R - 6 \cdot \sigma R \text{ otherwise} \end{array} \right)$$

$x0R(0.000) = \blacksquare$

2. Integration bounds assuming αE>0, αR arbitrary:

$$x0(\alpha E, \alpha R) := \begin{array}{|l} \max(x0E(\alpha E), x0R(\alpha R)) \text{ if } \alpha R \geq 0 \\ x0E(\alpha E) \text{ otherwise} \end{array}$$

$x0(0.608, 0.301) = \blacksquare$

$$x1(\alpha E, \alpha R) := \begin{array}{|l} \mu R + 6 \cdot \sigma R \text{ if } \alpha R \geq 0 \\ x0R(\alpha R) \text{ otherwise} \end{array}$$

$x0(0.608, -1) = \blacksquare$

$x1(1, -1) = \blacksquare$

3. Transformation to the standardised normal distribution $\Phi(u)$ (for any α):

Standardised variable E: $uE(x) := \frac{(x - \mu E)}{\sigma E}$ Transformed standardised variable E:

$$uuE(x, \alpha E) := \begin{array}{|l} \frac{\ln\left(\left|uE(x) + \frac{1}{CE(\alpha E)}\right|\right) + \ln\left(|CE(\alpha E)| \cdot \sqrt{1 + CE(\alpha E)^2}\right)}{\mathrm{sign}(\alpha E) \cdot \sqrt{\ln\left(1 + CE(\alpha E)^2\right)}} \text{ if } \alpha E \neq 0 \\ uE(x) \text{ otherwise} \end{array}$$

$$\phi E(x, \alpha E) := \begin{array}{|l} \frac{\mathrm{dnorm}(uuE(x, \alpha E), 0, 1)}{\sigma E \cdot \left|uE(x) + \frac{1}{CE(\alpha E)}\right| \cdot \sqrt{\ln\left(1 + CE(\alpha E)^2\right)}} \text{ if } CE(\alpha E) \neq 0 \\ \frac{\mathrm{dnorm}(uuE(x, \alpha E), 0, 1)}{\sigma E} \text{ otherwise} \end{array}$$

$\phi E(50, 0) = \blacksquare$

$\phi E(50, 0.0001) = \blacksquare$

Standardised variable R: $uR(x) := \frac{(x - \mu R)}{\sigma R}$ Transformed standardised variable R:

$$uuR(x,\alpha R) := \begin{cases} \dfrac{\ln\left(\left|uR(x) + \dfrac{1}{CR(\alpha R)}\right|\right) + \ln\left(|CR(\alpha R)|\cdot\sqrt{1+CR(\alpha R)^2}\right)}{\mathrm{sign}(\alpha R)\cdot\sqrt{\ln\left(1+CR(\alpha R)^2\right)}} & \text{if } \alpha R \neq 0 \\ uR(x) & \text{otherwise} \end{cases}$$

Distribution function $\Phi_{LN,X}(x) = \Phi_{LN,U}(u) = \Phi(uu)$:

$\Phi R(x,\alpha R) := \mathrm{pnorm}(uuR(x,\alpha R),0,1)$ $\quad uuR(50,0) = -5$ $\quad \Phi R(0,-0.3) = 1.237\times 10^{-12}$

4. Failure probability pf using transformation to normal distribution (for αE>0, αR arbitrary:

$$pf(\alpha E,\alpha R) := \int_{x0(\alpha E,\alpha R)}^{x1(\alpha E,\alpha R)} \phi E(x,\alpha E)\cdot\Phi R(x,\alpha R)\,dx \qquad pf(0.608,0.0001) = 8.745\times 10^{-4}$$

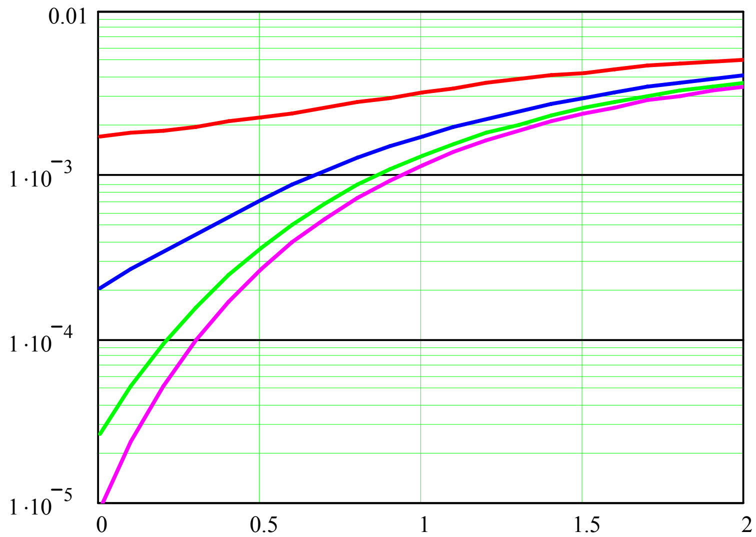


Figure 1. Failure probability pf=P{E>R} versus αE.

5. Alternative procedure for determination of failure probability using built-in distribution function for log-normal distributiopn $\Phi_{LN,X}(x)$ (for positive α only):

$$mE(\alpha E) := -\ln(|CE(\alpha E)|) + \ln(\sigma E) - (0.5)\cdot\ln\left(1+CE(\alpha E)^2\right) \qquad sE(\alpha E) := \sqrt{\ln\left(1+CE(\alpha E)^2\right)}$$

Probability density of E: $\phi E(x,\alpha E) := \mathrm{dlnorm}(x - x0E(\alpha E), mE(\alpha E), sE(\alpha E))$ $\quad \phi E(50,0.0001) = 0.04$

$$mR(\alpha R) := -\ln(|CR(\alpha R)|) + \ln(\sigma R) - (0.5)\cdot\ln\left(1+CR(\alpha R)^2\right) \qquad sR(\alpha R) := \sqrt{\ln\left(1+CR(\alpha R)^2\right)}$$

Distribution function of R: $\quad \Phi R(x,\alpha R) := \mathrm{plnorm}(x - x0R(\alpha R), mR(\alpha R), sR(\alpha R))$

Failure probability pf (for positive α only):

Check:

$$pf(\alpha E,\alpha R) := \int_{x0(\alpha E,\alpha R)}^{x1(\alpha E,\alpha R)} \phi E(x,\alpha E)\cdot\Phi R(x,\alpha R)\,dx \qquad pf(0.608,0.0001) = 8.745\times 10^{-4}$$

CHAPTER III - RELIABILITY DIFFERENTIATION

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Summary

Basic reliability elements specified in current standards for structural design commonly include failure probability related to a certain reference period T. Required reliability level of buildings and other civil engineering works is usually specified by the design (target) failure probability p_d or by appropriate reliability index β_d corresponding to a specified design working life T_d (for example 50 years). In reliability verification the design values β_d and T_d are sometimes replaced with an alternative reliability index β_a derived from the design values β_d and T_d for a convenient reference period T_a (for example 1 year).

Submitted study clarifies relationships between the alternative elements β_a, T_a and design values β_d, T_d, and indicates relevant procedures for reliability verification when alternative reference period T_a is considered. It is emphasised that verification based on β_a, T_a should be distinguished from verification of temporary or auxiliary structures when the design working life T_d itself is short. Theoretical consideration and numerical examples show that characteristic values and partial factors of basic variables describing material properties and self-weight are significantly dependent on the relevant reference period.

1 INTRODUCTION

1.1 Background documents

Recent documents [1], national [2], [3] and international documents ([4] to [7]) provide general principles and guidance for application of probabilistic methods to structural designs. The latest European document [5] and international standards [6] and [7] also indicate a theoretical basis of the so called "partial factor method" and procedures for determination of partial factors of material properties and actions using probabilistic principles.

The basic reliability elements considered in these procedures include probability of failure p (or equivalent reliability index β) corresponding to a certain reference period T used in verification of structural reliability. The reference period T used in verification may or may not coincide with the design working life T_d, which is the time period during which a structure is required to perform adequately. When the reference period used in reliability verification is different from T_d then it is called an alternative period and denoted in this study T_a.

1.2 General Principles

Basic probabilistic methods are used to analyse principles of reliability differentiation. Similarly as in Chapter I in this Handbook two essentially different cases are distinguished in the following:

- an alternative reference period T_a (for example 1 or 5 years), which is different from the design working life T_d (for example 50 years), is considered; this case is applicable when probabilistic models related to the period T_a are more credible than those related to T_d;

- the design working life T_d itself is short (for example 2, 5 or 10 years); this is the case of temporary or auxiliary structures and structures under a transient design situation (during execution or repair).

In the following the principles of reliability differentiation specified in current international documents [5,6,7] and related procedures for determining reliability measures to be applied in verification cases considering various design-working lives are discussed. Appropriate reliability elements (characteristic values and partial factors) are derived for material properties, self-weight and climatic actions (temperature, snow and wind) taking into account time dependence of failure probability and the reliability index.

2 BASIC RELIABILITY ELEMENTS

The basic reliability measures include the probability of failure and reliability index as introduced in Chapter I and II in this Handbook. The probability of structural failure P_f can be generally defined as

$$P_f = \mathrm{P}\{Z(\mathbf{X}) < 0\} \tag{1}$$

The limit state (performance) function $Z(\boldsymbol{X})$ is formulated in such a way that the reliable (safe) domain of a vector of basic variables $\boldsymbol{X} = X_1, X_2, \dots, X_n$ corresponds to the inequality $Z(\boldsymbol{X}) > 0$ while the failure domain to the complementary inequality $Z(\boldsymbol{X}) < 0$. A simple example of $Z(\boldsymbol{X})$ describes the basic relationship between the resulting load effect E and resistance R

$$Z(\boldsymbol{X}) = Z = R - E \tag{2}$$

The random variable Z in equation (2) is often called the reliability (safety) margin; its mean μ_Z, standard deviation σ_Z and skewness ω_Z may be derived from corresponding characteristics of resulting variables R and E as indicated in Chapter II.

Instead of the failure probability P_f, the reliability index β is frequently used in reliability consideration as an equivalent quantity to P_f. The reliability index β is related to the failure probability P_f as already indicated in Chapter I

$$P_f = \Phi(-\beta) \tag{3}$$

In this equation, $\Phi(\ )$ denotes the distribution function of standardised normal distribution. Note that, if the safety margin Z has normal distribution, then the reliability index may be determined simply as the ratio of μ_Z and σ_Z, thus $\beta = \mu_Z / \sigma_Z$ (in this case β denotes the distance of the mean μ_Z from the origin taking the standard deviation σ_Z as a unit). Chapter I shows the numerical relationship of both quantities. It should be emphasized that the failure probability P_f and the reliability index β represents fully the equivalent reliability measures with one to one mutual correspondence given by equation (3).

In the recent European document [5] a design working life for common structures is considered as T_d = 50 years, the reliability index for ultimate limit states β_d = 3,8 corresponds to the design failure probability $P_d = 7{,}2 \times 10^{-5}$, for serviceability limit states β_d = 1,5 and $p_d = 6{,}7 \times 10^{-2}$ (a more appropriate term is the “target probabilities” used in ISO documents [6] and [7]). These quantities are recommended as reasonable minimum requirements and it is emphasized that P_d and β_d are formal conventional quantities only and may not correspond to actual frequency of failures.

In design analysis of a structure it is generally required that

$$P_f \le P_d \tag{4}$$

or equivalently in terms of reliability index

$$\beta \geq \beta_d \tag{5}$$

where p_d denotes specified design (target) failure probability corresponding to the target reliability index β_d.

Conditions (4) or (5) have to be used by designers when probabilistic methods are applied for verification of structural reliability. Indicative target values p_d and β_d are declared in some national standards (e.g. [2] and [3]) and recently also specified in international documents (e.g. [4] to [7]) for various design conditions (limit states, failure consequences and economic aspects).

3 DESIGN WORKING LIFE AND RELIABILITY

Design working life T_d is an assumed period of time for which a structure or part of it is to be used for its intended purpose with anticipated maintenance but without major repair being necessary. In recent documents of CEN [5] and ISO [6] indicative values of T_d are provided for five categories of structures as shown in Chapter I of this Handbook.

More detailed specification of structural categories and design working lives is available in the ISO documents [6, 7]. In general the design working lives indicated in [2] are greater (in some cases by 100 %) than those given in Chapter I. For example the design working life for temporary structures indicated in [2] is 15 years, for agricultural structures 50 years, for apartment and office buildings 100 years, and for railway structures, dams, tunnels and other underground engineering works 120 years.

Design failure probabilities p_d are usually indicated in relation to the expected social and economical consequences. Table 1 shows classification of target reliability levels provided in EN 1990 [5]. Reliability indexes β are given for two reference periods T (1 year and 50 years) only, without any explicit link to the design working life T_d. Similar β-values as in Table 1 are given in [3] for the ultimate limit states, for which, however, the design working life T_d = 80 years (for building structures) is considered.

It should be underlined that a couple of β values (β_a and β_d) specified in Table 1 for each reliability class (for 1 year and 50 years) correspond to the same reliability level. Practical application of these values depends on the time period T_a considered in the verification, which may be connected with available information concerning time variant vector of basic variables $\boldsymbol{X} = X_1, X_2, ..., X_n$. For example, if the reliability class 2 and 50 years design working period is considered, then the reliability index β_d = 3,8 should be used in the verification of structural reliability. The same reliability level corresponding to class 2 is achieved when the time period T_a = 1 year and β_a = 4,7 is used. Thus, various reference periods T_a, in general different from the design working life T_d, may be used for achieving a certain reliability level.

Table 1. Reliability classification in accordance with CEN [5]

Reliability classes	Consequences for loss of human life, economical, social and environmental consequences	Reliability index β		Examples of buildings and civil engineering works
		β_a for T_a= 1 year	β_d for T_d= 50 years	
3 – high	High	5,2	4,3	Bridges, public buildings
2 – normal	Medium	4,7	3,8	Residential and office buildings
1 – low	Low	4,2	3,3	Agricultural buildings, greenhouses

Similar target β_d values are provided in ISO 2394 [6] for the design working life T_d (called in ISO "life time") without specification of any particular value of T_d. As indicated in Table 2, two factors are considered for reliability differentiation in [6]: relative costs of safety measures and consequences of failure.

Table 2. Target reliability index β_d for the design working life T_d given in ISO 2394 [6]

Relative costs of safety measures	Consequences of failure small	some	moderate	great
High	0	1,5	2,3	3,1
Moderate	1,3	2,3	3,1	3,8
Low	2,3	3,1	3,8	4,3

It appears that available documents do not provide an explicit guidance on how to take into account the design working life T_d. Both international documents CEN [5] and ISO [6] give the target value β_d for specific reference periods T, however, no explicit rule is offered for adjustment of target value β_d to different working design lives T_d recommended for various types of construction works.

Nevertheless, some indication is provided in another ISO document [7] for assessment of existing structures where it is recommended that reliability levels for any residual lifetime could be similar to those considered for the design working life T_d in the case of a new structure. Consequently, similar reliability levels (expressed in terms of probability p_d or reliability index β_d) may be considered when designing structures for different design working lives T_d, for example for $T_d = 50$ and $T_d = 25$ years.

4 VARIATION OF FAILURE PROBABILITY WITH TIME

When the vector of basic variables $\boldsymbol{X} = X_1, X_2, \ldots, X_m$ is time variant, then failure probability p is also time variant and should always be related to a certain reference period T, which may be generally different from the design working life T_d. Considering a structure of a given reliability level, the design failure probability $p_d = p_n$ related to a reference period $T_n = n\, T_1$ can be derived from the alternative probability $p_a = p_1$ corresponding to $T_a = T_1$ (to simplify notation note that previously used subscript "d" corresponds now to "n" and subscript "a" to "1") using approximate relationship given in [6], [7]

$$P_n = 1 - (1 - P_1)^n \tag{6}$$

For very small probabilities, this relationship could be further simplified as $p_n = p_1\, T_n / T_1$. Time periods T_1 and T_n may have an arbitrary length and $n = T_n / T_1$ may not be an integer; T_1 is, however, often one year. Probability p_n increases (almost linearly) with T_n.

It follows from equation (6) that reliability indexes $\beta_1 = \beta_a$ and $\beta_n = \beta_d$, given in accordance to equation (3) as $p_1 = \Phi(-\beta_1)$ and $p_n = \Phi(-\beta_n)$ are related as follows [5]

$$\Phi(\beta_n) = [\Phi(\beta_1)]^n \tag{7}$$

Here $\Phi(.)$ denotes the distribution function of standardised normal distribution. Figure 1 shows variation of β_n with β_1 for n = 5, 25, 50 and 100. Note that, if the reference period T_1 is one year, then n indicates the number of years of the reference period T_n ($n = T_n$).

Figure 1 confirms data indicated in Table 1. For example, if the target reliability level of a structure is specified by β_{50} = 3,8 for the design working life $T_d = T_n$ = 50 years, then it could be verified using reference period $T_a = T_1$ =1 year and $\beta_a = \beta_1$ =4,7. When, however, the same reliability index 3,8 is specified for a structure having a design working life T_n = 25 years only, thus β_{25} = 3,8, then the reliability of this structure could be verified using an alternative reference period T_1 = 1 year and reliability index β_1 = 4,5, similarly when β_5 = 3,8 then β_1 = 4,2 (see Figure 1).

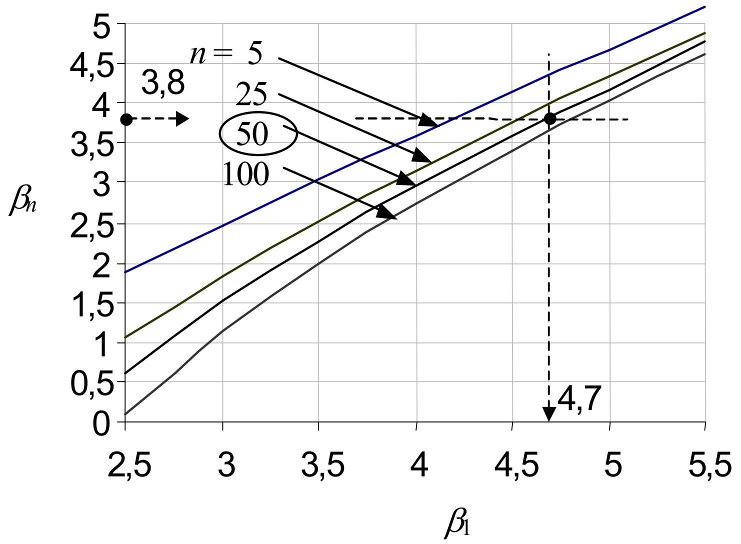

Figure 1. Variation of β_n with β_1 for n = 5, 25, 50 and 100

Note that, if 1-year period would be used for specification of the target reliability level of a structure, then Figure 1 provides information on the resulting failure probability corresponding to a given working life T_n. For example, if the target reliability level is specified by the reliability index β_1 = 4,7 (corresponding to the probability $p_1 = 1,3 \times 10^{-6}$), then (as already mentioned) the reliability level of a structure having a working life, T_n = 50 years is characterised by β_{50} = 3,8. Similarly when a period T_n = 5 years is used, then β_5 = 4,3 or when T_n = 100 years, then β_{100} = 3,6.

So, the reliability level of a structure can be specified using different time periods T, which may not necessarily coincide with the design working life T_d. This may be useful when experimental data concerning time variant basic variables are available for a specific reference period T (for example 1 or 5 years) that is different from the design working life T_d. In such a case, however, all the basic variables (including those that are time independent) should be considered by appropriate design values related to the same reference period T. The following simple example indicates the effect of using a reference period T different from the design working life considering a resistance variable (strength) having lognormal distribution.

5 PARTIAL FACTOR OF A MATERIAL PROPERTY

Consider a resistance variable R (strength) having lognormal distribution. When an alternative reference period T_a instead of the design working life T_d is used in reliability verification of a structure, then the design value of R should be determined for T_a instead of T_d. It is assumed that the characteristic value R_k of R is defined as its 5% fractile [5], [6] a [7].

Then of the resistance variable R, the characteristic value R_k and design value R_d are defined as [4], [5]

$$R_k = \mu_R \times \exp(-1{,}645 \times V_R) \tag{8}$$

$$R_d = \mu_R \times \exp(-\alpha_R \times \beta_a \times V_R) \tag{9}$$

Taking into account equations (8) and (9) it follows that the partial factor is given as

$$\gamma_R = R_k / R_d = \exp(-1{,}645 \times V_R) / \exp(-\alpha_R \times \beta_a \times V_R) \tag{10}$$

Considering selected values of the coefficient of variation V_R, Figure 2 shows the partial factor γ_R for lognormal distribution of R (equation (10)).

It follows from Figure 2 that when reliability of a structure is verified using a short alternative reference period T_a (for example for example for T_a = 1 year when β_a = 4,7), the partial factor γ_R should generally be greater than in the case when the whole design working life T_d (for example for T_d = 50 when β_d = 3,8) is considered. It may be noted that the partial factor γ_R of material property R increases with the increasing value of the reliability index β_a.

Similar conclusions can be expected for partial factors of other basic variables, in particular for partial factors of permanent actions.

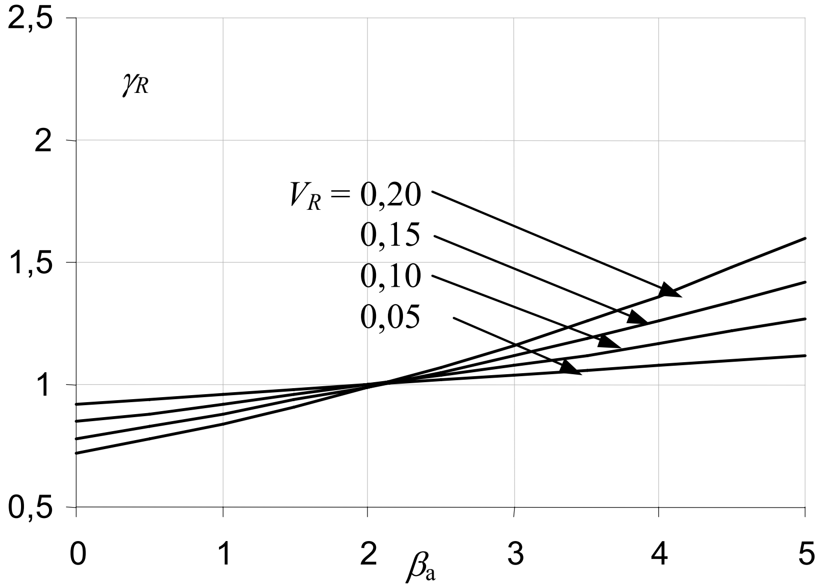

Figure 2. Variation of γ_R with β_a for selected coefficients of variation (R lognormal)

6 PARTIAL FACTORS OF SELF-WEIGHT

Consider a self-weight G having normal distribution. Similarly as in the case of material property, when an alternative reference period T_a instead of the design working life T_d is used in reliability verification of a structure, then the design value of G should be determined for T_a instead of T_d. The characteristic value G_k of G is defined as the mean μ_G [5], [6] and [7]:

$$G_k = \mu_G \tag{11}$$

The design value G_d is given as [4], [5]

$$G_d = \mu_G - \alpha_G \times \beta \times \sigma_G = \mu_G + 0{,}7\times \beta_a \times \sigma_G = \mu_G(1 + 0{,}7\times \beta_a \times V_G) \tag{12}$$

In equation (11) and (12) μ_G denotes the mean, σ_G the standard deviation, V_G the coefficient of variation and $\alpha_G = -0{,}7$ the sensitivity factor of G. The partial factor γ_G of G is defined as [5], [6] a [7]

$$\gamma_G = G_d / G_k \tag{13}$$

Taking into account equations (11) and (12) it follows from (13) that

$$\gamma_G = (1 + 0{,}7\times \beta_a \times V_G) \tag{14}$$

Figure 3 shows variation of the partial factor γ_G with the reliability index β_a for selected values of the coefficient of variation V_G = 0,05; 0,10; 0,15 and 0,20. Note that γ_G = 1,35 (recommended in EN 1990 [5]) corresponds approximately to the reliability index β_a = 3,8 if the coefficient of variation is about 0,1 (the value in EN 1990 [5] was increased by 5% to take into account model uncertainty).

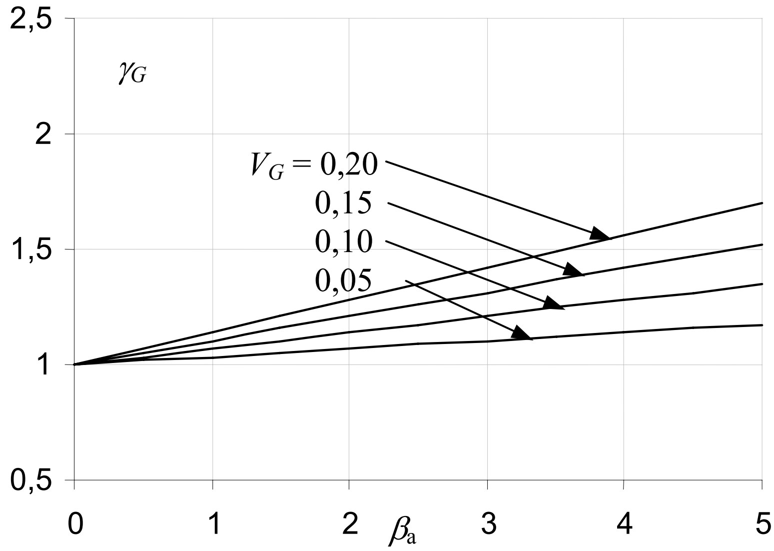

Figure 3. Variation of γ_G with β_a and coefficient of variation V_G (G normal).

Assuming the coefficient of variation 0,1 for both the resistance R and the self weight G Figures 2 and 3 indicate that the partial factor of self-weight γ_G varies slightly more significantly with β_a - values than with the partial factor γ_R of resistance variable R. This finding is, however, dependent on the distributions assumed for both variables.

7 CLIMATIC ACTIONS AND IMPOSED LOADS

Drafts of European documents for climatic actions due to temperature [8], snow [9] and wind [10] indicate possible reduction of characteristic values Q_k for temperature, snow load and wind speed in case of shorter reference (return) period (for example 5 years) than 50 years considered in normal cases. Such a reduction may be applied in transient design situations (for example during execution).

The following relationships for thermal, snow and wind actions, respectively, are recommended in relevant Parts of Eurocode EN 1991:

(a) In accordance with EN 1991-1-5 [8] Thermal actions, the maximum and minimum shade air temperature $T_{\mathrm{max},50}/T_{\mathrm{min},50}$ for 50-year return period may be reduced to $T_{\mathrm{max},n}/\ T_{\mathrm{min},n}$ for n-year return period using the following formulae

$$T_{\mathrm{max},n} = k\ T_{\mathrm{max},50}\text{ , for } k = \{k_1 - k_2 \ln[-\ln(1\text{-}1/n)]\} \tag{15}$$

$$T_{\mathrm{min},n} = k\ T_{\mathrm{min},50},\ \text{ for } k = \{k_3 + k_4 \ln[-\ln(1\text{-}1/n)]\} \tag{16}$$

where $T_{\mathrm{max},n}/T_{\mathrm{min},n}$ is the maximum/minimum, and the coefficients k_1 = 0,781, k_2 = 0,056, k_3 = 0,393, k_4 = -0,156 might be used (based on data of UK [11]),

(b) In accordance with EN 1991-1-3 [9] Snow actions the characteristic value of snow action $s_{\mathrm{k},n}$ corresponding to the return period of n years is given using Gumbel distribution as

$$s_{\mathrm{k},n} = k\ s_{\mathrm{k},50},\text{ where } k = \frac{1 - V_s \dfrac{\sqrt{6}}{\pi}\left[\ln(-\ln(1-p)) + 0{,}57722\right]}{1 - V_s \dfrac{\sqrt{6}}{\pi}\left[\ln(-\ln(0{,}98)) + 0{,}57722\right]} \tag{17}$$

where $s_{\mathrm{k},50}$ is the characteristic snow load on the ground for 50-year return period and $s_{\mathrm{k,n}}$ for n-year return period, p denotes here the probability of $s_{\mathrm{k,n}}$ being exceeded corresponding to n years of return period and V_{s} is the coefficient of variation of annual maximum snow load,

(c) In accordance with EN 1991-1-4 [10] the basic wind speed $v_{\mathrm{b},n}$ having the return period n years may be assessed using semi-empirical expressions

$$v_{\mathrm{b},n} = k\ v_{\mathrm{b},50},\text{ where } k\ = \left[\frac{1 - K\ln(-\ln(1-p))}{1 - K\ln(-\ln(0{,}98))}\right]^{0{,}5} \tag{18}$$

where $v_{\mathrm{b},50}$ is the basic wind velocity for 50-year return period and $v_{\mathrm{b},n}$ for n-year return period and p denotes here the probability of $v_{\mathrm{b},n}$ being exceeded corresponding to n years of return period. The constant K in equation (18) follows from Gumbel distribution as $K = V_v\sqrt{6}/\pi$, where V_v denotes coefficient of variation of annual wind speed. An approximate value K = 0,2 (which corresponds to the coefficient of variation V_v= 0,26) is used in the following comparison of reduction coefficients k for considered climatic actions.

Table 3 shows reduction coefficients k for climatic actions (applied in a general relationship $Q_{\mathrm{k},n} = k\ Q_{\mathrm{k},50}$) for selected return periods of n - years.

Table 3. Reduction coefficient k for climatic actions ($Q_{\mathrm{k},n}$ = k $Q_{\mathrm{k},50}$) for different return periods of n - years.

Return period of n-years	p	Reduction coefficient k for			
		$T_{\mathrm{max},n}$	$T_{\mathrm{min},n}$	$s_{\mathrm{n},n}$	$v_{\mathrm{b},n}$
2 years	0,5	0,8	0,45	0,64	0,77
5 years	0,2	0,86	0,63	0,75	0,85
10 years	0,1	0,91	0,74	0,83	0,90
50 years	0,02	1	1	1	1

It follows from Table 3 that the characteristic value of climatic actions may be considerably reduced if shorter reference period is considered in the design. For example for 5-year return period of action due to snow or wind reduces to 75 or 85 % of the characteristic values for 50-year return period, similarly the characteristic value of the maximum shade air temperature to 86%, the minimum shade air temperature even to 63%. Note that in verification of bridge decks during execution phases the characteristic values of uniform temperature components are derived from shade air temperature [8].

It should be noted that no reduction of partial factors for load is indicated in documents [8], [9] and [10]. Thus, the same reliability level as for 50-year design working life described by $p_{\mathrm{d}} = 7{,}2 \times 10^{-5}$ ($\beta_{\mathrm{d}} = 3{,}8$) may be considered also for the reference period $T = n$ years. Certainly, a different reliability level (for example reduced to $\beta_{\mathrm{d}} < 3{,}8$) can be chosen taking into account economic and other aspects in accordance to the principles of reliability differentiation discussed above.

Imposed load could be possibly also reduced when short reference time is considered similarly as climatic actions. Some statistical data are available in documents of JCSS [12]. However, a variety of random properties of different types of imposed loads make it very difficult to formulate general rules. Unless convincing data are available the characteristic values specified in current documents may be accepted without any reduction.

8 EXAMPLES

Consider a steel structure having the design working life $T_{\mathrm{d}} = 50$ years, for which the target failure probability is specified as $p_{\mathrm{d}} = 7{,}2 \times 10^{-5}$ ($\beta_{\mathrm{d}} = 3{,}8$). Failure probability p for the alternative reference period $T_{\mathrm{a}} = 1$ year, which is considered in design due to data concerning actions, will be lower than the target failure probability p_{d} ($p < p_{\mathrm{d}}$ and $\beta > \beta_{\mathrm{d}}$); from equation (6):

$$p_{\mathrm{a}} = 1 - (1\text{-}7{,}2 \times 10^{-5})^{1/50} = 1{,}44 \times 10^{-6}$$

When the reference period $T_{\mathrm{a}} = T_1 = 1$ year is considered in design verification, then the reliability index β follows from equation (7) as

$$\beta_1 = -\,\Phi^{-1}(1{,}44 \times 10^{-6}) = 4{,}7$$

Reliability index β_1 is greater than the target value $\beta_{\mathrm{d}} = 3{,}8$ specified for the design working life $T_{\mathrm{d}} = 50$ years.

Using equation (10) the partial safety factor γ_R for $T_{\mathrm{a}} = T_1 = 1$ year assuming the coefficient of variation $V_R = 0{,}08$ (corresponding to the common variability of strength of structural steel) the partial safety factor is given as (see also Figure 2)

$$\gamma_R = \exp\,(-\,1{,}645 \times 0{,}08) / \exp\,(-\,0{,}8 \times 4{,}7 \times 0{,}08) = 1{,}18$$

Note that when the design working life $T_{\mathrm{d}} = 50$ is considered in reliability verification then:

$$\gamma_R = \exp\,(-\,1{,}645 \times 0{,}08) / \exp\,(-\,0{,}8 \times 3{,}8 \times 0{,}08) = 1{,}12$$

Obviously, the partial factor γ_R increases with the decreasing reference period T_{a}.

The partial factor of self-weight γ_G is given by equation (14). Assume again, that the specified reliability level for 50-year design working life is given by $\beta_{\mathrm{d}} = 3{,}8$. Assuming the coefficient of variation $V_G = 0{,}1$ and considering the one year time period for reliability verification ($\beta_1 = \beta_{\mathrm{a}} = 4{,}7$), then the partial factor γ_G that should be used is

$$\gamma_G = (1 + 0{,}7 \times 4{,}7 \times 0{,}1) = 1{,}33$$

If the verification period is equal to the design working life ($\beta_d = \beta_a = \beta_{50} = 3{,}8$), then

$$\gamma_G = (1 + 0{,}7 \times 3{,}8 \times 0{,}1) = 1{,}27$$

Thus, the variation in γ_G is less significant than the variation in γ_R (see also Figure 3).

A different task is reliability verification of an agricultural structure having the design working life $T_d = 25$ years, for which the target reliability index can be decreased to $\beta_d = 3{,}3$ (see Table 1). It follows from equation (10) that the partial factor γ_R for $T_d = 25$ is

$$\gamma_R = \exp\,(-\,1{,}645 \times 0{,}08) / \exp\,(-\,0{,}8 \times 3{,}3 \times 0{,}08) = 1{,}08$$

The partial factor γ_R may, therefore, be decreased from 1,15 to about 1,1. However it should be emphasized that this reduction of γ_R is due to a reduced target reliability index β_d = 3,3 and not due to a shorter design working life $T_d = 25$ instead of the usual $T_d = 50$ years.

Annex A includes MATHCAD Sheet “GammaRG” that can be used to make numerical calculations.

9 CONCLUDING REMARKS

(1) In present international documents the target values of failure are related to economic aspects of safety measures and consequences of structural failure only vaguely, without any explicit relation to various design working lives T_d for different types of structures.

(2) When alternative failure probability p_a is derived for a suitable reference period T_a from the target failure probability p_d and design working life T_d, partial factors and characteristic values of variable actions for p_a and T_a should also be specified.

(3) For temporary structures, with a short design working life T_d, the target failure probability p_d can be specified in accordance with the general principles of reliability differentiation; reliability elements for basic variables should be derived for specified p_d and T_d.

(4) The partial factors γ derived for an alternative reference period T_a different from T_d may vary considerably from the values corresponding to the design working life T_d depending on T_a and distributions of relevant basic variables.

(5) The partial factor of self-weight γ_G corresponding to an alternative reference period T_a varies with β_a-values less significantly than the partial factor of material property γ_R.

(6) Partial factors γ_R derived for an alternative reference period T_a of one-year may be considerably greater than γ_R specified for the design working life T_d.

(7) Following recommendations of Eurocodes, the characteristic value for climatic actions due to snow corresponding to 5-year return (reference) period may be reduced to 75 % of the characteristic values for 50-year return period, similarly the characteristic value of wind speed may be reduced to 85 %, the maximum temperature to 86%, the minimum temperature to 63%.

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[4] ENV 1991-1 Basis of design and actions on structures. Part 1: Basis of design. CEN 1994.
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[9] EN 1991-1-3 Eurocode 1 Actions on structures. Part 1-3: Snow actions. European Committee for Standardisation, 06/2002.
[10] EN 1991 Actions on structures. Part 1-4: Wind load. European Committee for Standardisation, 06/2002.
[11] G. König et al: New European Code for Thermal Actions, Background document, Report N.6, University of Pisa, 1999.
[12] JCSS: *Probabilistic model code.* JCSS working materials, http://www.jcss.ethz.ch/, 2001.

ATTACHMENTS

1. MATHCAD sheet "GammaRG.mcd"

MATHCAD sheet Gamma is intended for determination of the partial factor γ_R of the resistance *R* and the partial factor γ_G of the permanent load *G*.

2. MATHCAD sheet "PSI0.mcd"

MATHCAD sheet PSI0 is intended for determination of the Combination factor ψ0 for accompanying action.

3. MATHCAD sheet "PSI12.mcd"

MATHCAD sheet PSI12 is intended for determination of the combination factor ψ12 for accompanying action.

Attachment 1 - MATHCAD sheet "Gammarg.mcd"

GammaR, gammaG for a theoretical model

MATHCAD sheet for determination of the charactreistic, design values and partial factors γR and γG.

Coeficients of fractile estimation given in EN 1990

5% fractile V uknown $k := 1.65$

0,1 % fractile V unknown $d := 3.09$

Sensitivity factors: $\alpha R := 0.8$ $\alpha E := -0.7$ $\beta R(\beta) := \beta \cdot \alpha R$ $\beta E(\beta) := \beta \cdot \alpha E$

Prameters: $\beta := 0, 0.1 .. 5$ $V := 0, 0.1 .. 0.5$

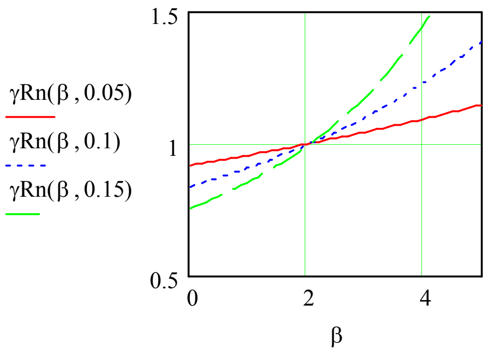
Characteristic and design values (relative values related to the mean) $xk=\xi k\sigma^{*}\mu x$, $xd=\xi ds*\mu x$

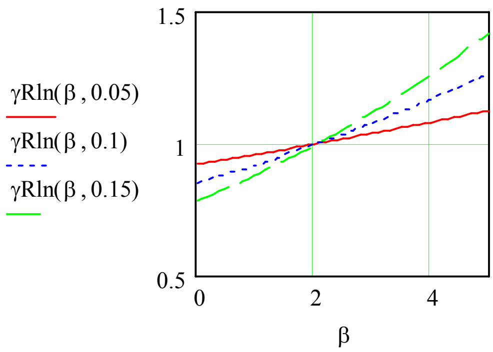
Normal distribution $\xi kn(V) := (1 - k \cdot V)$ $\xi dn(\beta, V) := (1 - \beta R(\beta) \cdot V)$ $\boxed{\xi dn(3.8, 0.1) = 0.696}$

Normal distribution $\xi kln(V) := \dfrac{\exp\left[(-k)\cdot\sqrt{\ln(1+V^2)}\right]}{\sqrt{1+V^2}}$ $\xi dln(V) := \dfrac{\exp\left[(-d)\cdot\sqrt{\ln(1+V^2)}\right]}{\sqrt{1+V^2}}$

$\xi dln(\beta, V) := \dfrac{\exp\left[(-\beta R(\beta))\cdot\sqrt{\ln(1+V^2)}\right]}{\sqrt{1+V^2}}$ $\boxed{\xi kn(0.1) = 0.835}$ $\boxed{\xi dln(3.8, 0.1) = 0.735}$

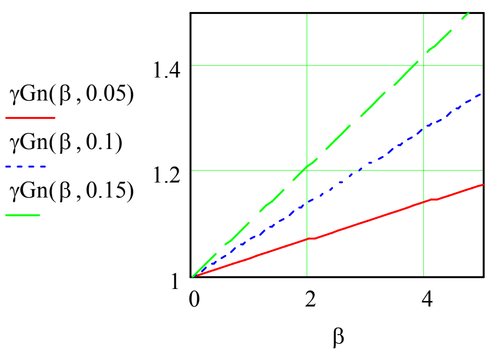
GammaR $\gamma Rn(\beta, V) := \dfrac{\xi kn(V)}{\xi dn(\beta, V)}$ $\gamma Rln(\beta, V) := \dfrac{\xi kln(V)}{\xi dln(\beta, V)}$ $\gamma Rn(3.8, 0.1) = 1.2$

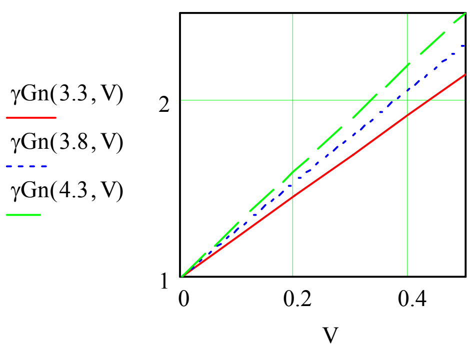




GammaG for permanent load assuming normal distribution

$\gamma Gn(\beta, V) := 1 - \beta E(\beta) \cdot V$ $\gamma Rn(3.8, 0.1) = 1.2$





Attachment 2 - MATHCAD sheet "PSI0.MCD"

MATHCAD sheet "PSI0" for calculating PSI 0 assuming theoretical models

Combination factor ψ_0 for accompanying action

1 Input data V is coefficient of variability of the accompanying action related to the reference period T (50 years), r = T/T1 where T1 is the greater of the of basic periods actions to be combined (for example 5, 7, 10, 50)

Range variables $V := 0.0, 0.05 .. 1.0$ $r := 1 .. 50$ $\beta := 3.8$ (reliability index)

2 Factor ψ_0 for normal distribution:

Formula following Turkstra's rule $\psi 0 = F^{-1}(\Phi(0{,}4{*}0{,}7\ \beta)^r)/F^{-1}(\Phi(0{,}7\ \beta))$:

$$\psi 0n(V,r) := \frac{1 + \mathrm{qnorm}\left(\mathrm{pnorm}(0.28\cdot\beta, 0, 1)^r, 0, 1\right)\cdot V}{1 + 0.7\beta\cdot V}$$

Check: $\boxed{\psi 0n(0.15, 7) = 0.67}$

Approximation in EC 1990

$$\psi 0na(V,r) := \frac{1 + (0.28\cdot\beta - 0.7\cdot\ln(r))V}{1 + 0.7\beta\cdot V}$$

$\boxed{\psi 0na(0.15, 7) = 0.683}$

3 Factor ψ_0 for Gumbel distribution:

$$\psi 0g(V,r) := \frac{1 - 0.78\cdot V\cdot(0.58 + \ln(-\ln(\mathrm{pnorm}(0.28\cdot\beta, 0, 1))) + \ln(r))}{1 - 0.78\cdot V\cdot(0.58 + \ln(-\ln(\mathrm{pnorm}(0.7\cdot\beta, 0, 1))))}$$

$\boxed{\psi 0g(0.15, 7) = 0.584}$

4 General $\psi 0 = F^{-1}(\Phi(0{,}4{*}0{,}7\ \beta c)^r)/F^{-1}(\Phi(0{,}7\ \beta c)^r)$:

$$\beta c(r) := -\mathrm{qnorm}\left(\frac{\mathrm{pnorm}(-0.7\cdot\beta, 0, 1)}{r}, 0, 1\right)$$

$$\psi 0d(V,r) := \frac{\mathrm{qgamma}\left[\left(\mathrm{pnorm}(0.4\cdot\beta c(r), 0, 1)^r\right), V^{-2}\right]}{\mathrm{qgamma}\left(\mathrm{pnorm}(\beta c(r), 0, 1)^r, V^{-2}\right)}$$

$\psi 0d(0.3, 7) = 0.488$ $\boxed{\beta c(7) = 3.259}$

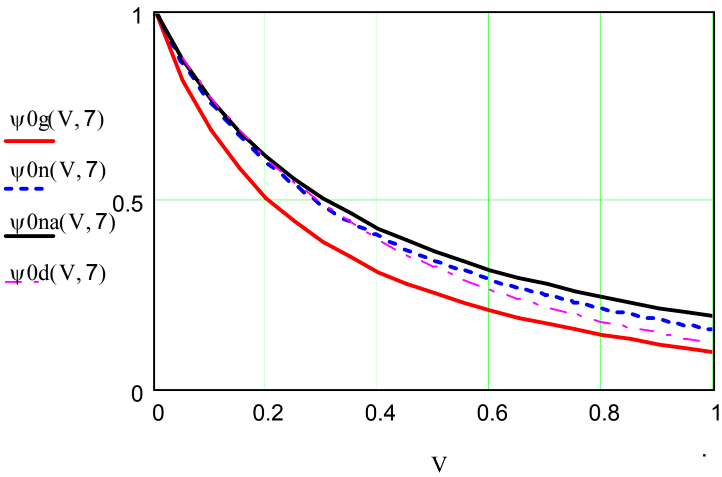


Figure 1. Variation of ψ_0 with V for selected distributions.

Check: $V := 0.1, 0.2 .. 0.5$

V =	$\psi 0n(V,10)$ =	$\psi 0na(V,10)$ =	$\psi 0g(V,10)$ =
0.1	0.727	0.747	0.664
0.2	0.548	0.581	0.474
0.3	0.423	0.465	0.352
0.4	0.33	0.378	0.268
0.5	0.258	0.312	0.205

Note. Gumbel distribution leads to the lowest $\psi 0$

4. Combination factor $\psi 0$, for time sensitivity factor αT < 1, V refers to period T (50 years)

$N := 10 \qquad \alpha T := 0.5 \qquad V := 0, 0.1 .. 1$

$$\psi 0g(\beta, V, N, \alpha T) := \frac{1 - 0.78 \cdot V \cdot (0.58 + \ln(-\ln(\mathrm{pnorm}(0.28\beta, 0, 1))) + \alpha T \ln(N))}{1 - 0.78 \cdot V \cdot (0.58 + \ln(-\ln(\mathrm{pnorm}(0.7\beta, 0, 1))))}$$

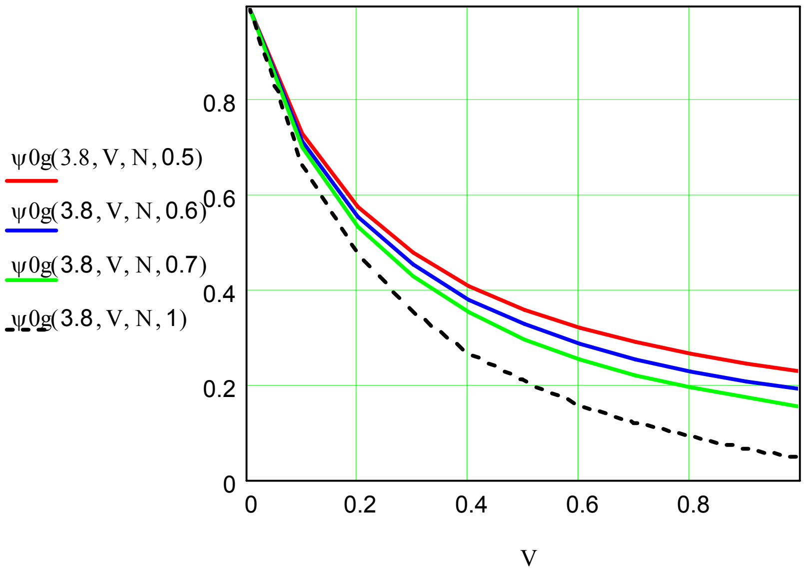


Figure 2. Variation of ψ_0 with *V* for Gumbel distribution and factors αT.

Variation of ψQ with αT

$N := 7 \qquad \alpha T := 0, 0.1 .. 1 \qquad V := 0.3$

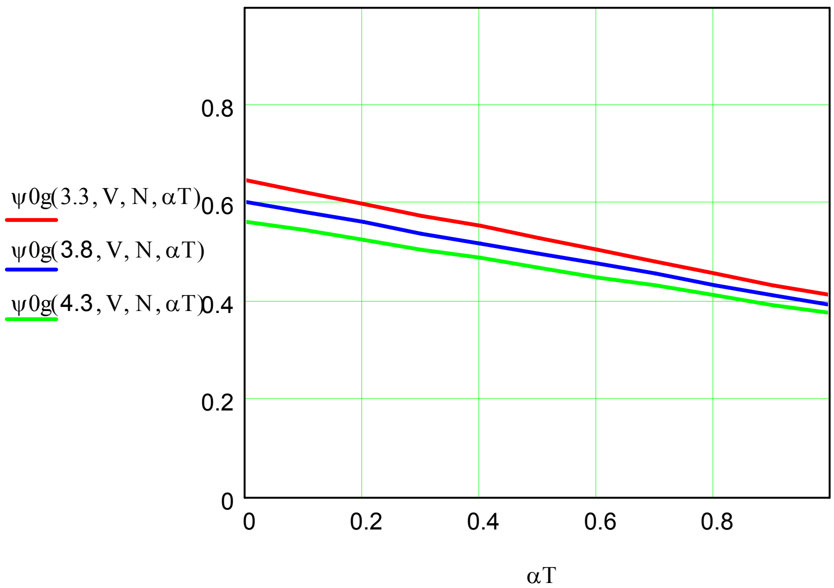


Figure 3. Variation of ψ_0 with αT for Gumbel distribution and reliability indeces β .

Attachment 3 - MATHCAD sheet "PSI12.MCD"

MATHCAD SHEET PSI12

Combination factor $\psi 12$ for accompanying action

1 Input data V is coefficient of variability of the accompanying action related to annual extremes.

Coefficient of variation referred to point in time distribution $w := 0, 0.1 .. 1.1$

Probability $\rho = 1 - \eta / q$ for determining Q1 $\rho := 0, 0.02 .. 1.01$

Rreliability index $\beta := 3.8$

2. Factor $\psi 12$ for normal distribution

$$\psi 12(w, \rho) := \frac{1 + \mathrm{qnorm}(\rho, 0, 1) \cdot w}{1 + \mathrm{qnorm}(0.98, 0, 1) \cdot w} \qquad \boxed{\psi 12(0.5, 0.5) = 0.493}$$

3 Factor $\psi 12$ for Gumbel distribution:

$$\psi 12g(w, \rho) := \frac{1 - 0.78 \cdot w \cdot (0.58 + \ln(-\ln(\rho)))}{1 - 0.78 \cdot w \cdot (0.58 + \ln(-\ln(0.98)))} \qquad \boxed{\psi 12g(0.5, 0.5) = 0.399}$$

4 Comparison of $\psi 12$ for normal and Gumbel distribution:

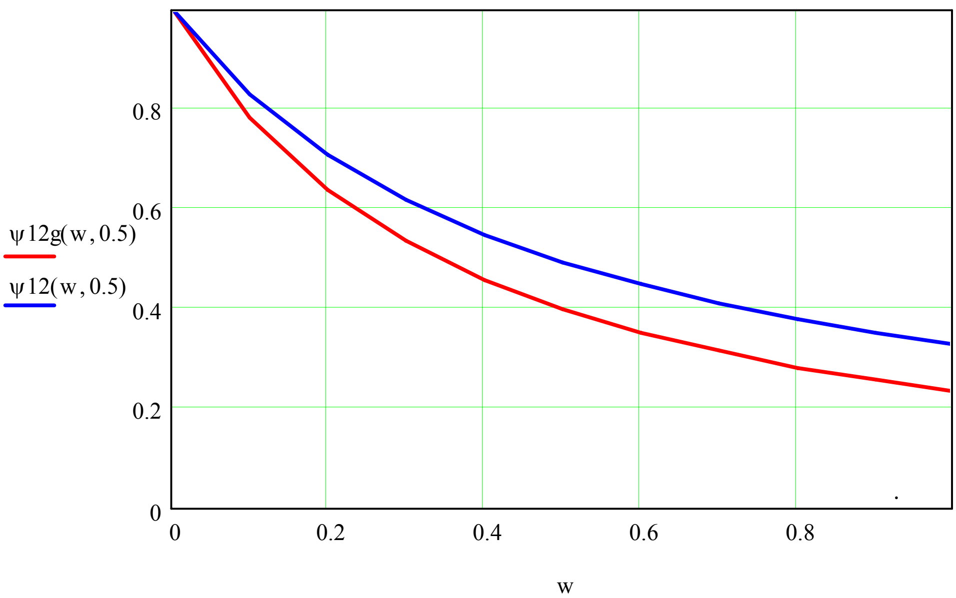


Figure 1. Variation of $\psi 12$ factor with the coefficient of variation V assuming the normal and Gumbel distribution

The Gumbel distribution leads to a lower $\psi 12$ factor than the normal distribution

5 Variation of $\psi 12$ with the probability ρ assuming Gumbel distribution:

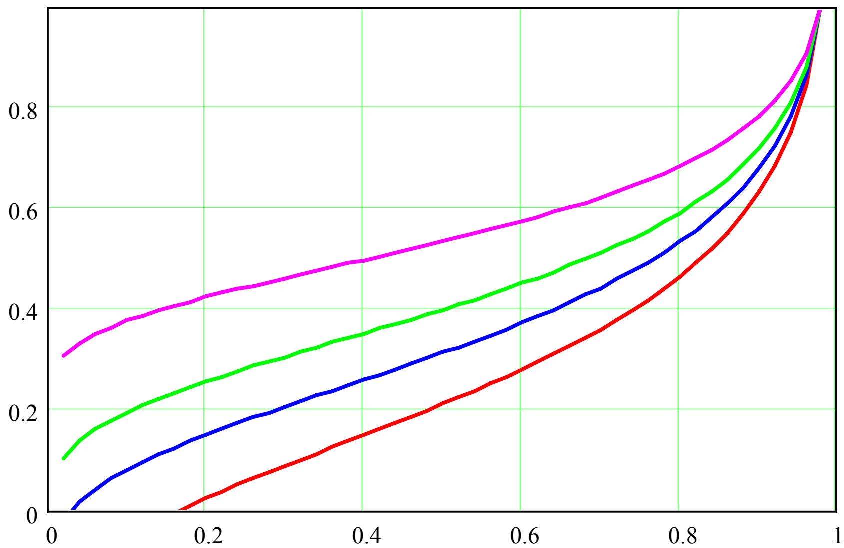


Figure 2. Variation of $\psi 12$ with the probability ρ for selected coefficients of variation V assuming Gumbel distribution.

Examples of probability $\rho = 1 - \eta / q$ where η is fraction of the refence period (0.01 or 0.5) during which Q1 and Q2 are exceeded, p probabilioty of Q being non zero

- imposed w=1.1,
  - short term $\psi 1$: on 18 days a year $\rho = 1 - \eta / q$ = 1- 0.01/0.05 ~ 0.8
  - long term $\psi 2$: almost always on $\rho = 1 - \eta / q$ = 1- 0.5/1 ~ 0.5
- wind w=0.5, $\psi 1$: 10x8 hours a year $\rho = 1 - \eta / q$ = 1- 0.01/0.009 ~ 0.1
  $\rho = 1 - \eta / q$ = 1- 0.5/0.009 ~ NA >> 0 $\psi 2$ ~ 0.0
- snow w=0.7, on 5 days a year $\rho = 1 - \eta / q$ = 1- 0.01/0.014 ~ 0.3 $\psi 1$ ~ 0.2
  $\rho = 1 - \eta / q$ = 1- 0.5/0.014 ~ NA >> 0 $\psi 2$ ~ 0.0

5 Variation of $\psi 12$ with the probability η and q assuming Gumbel distribution:

Probability of Q being non zero $q := 0, 0.01 .. 1$

$$\psi 12g(w, \eta, q) := \frac{1 - 0.78 \cdot w \cdot \left(0.58 + \ln\left(-\ln\left(1 - \frac{\eta}{q}\right)\right)\right)}{1 - 0.78 \cdot w \cdot (0.58 + \ln(-\ln(0.98)))} \qquad \psi 12g(0.5, 0.5, 0.50001) = -0.067$$

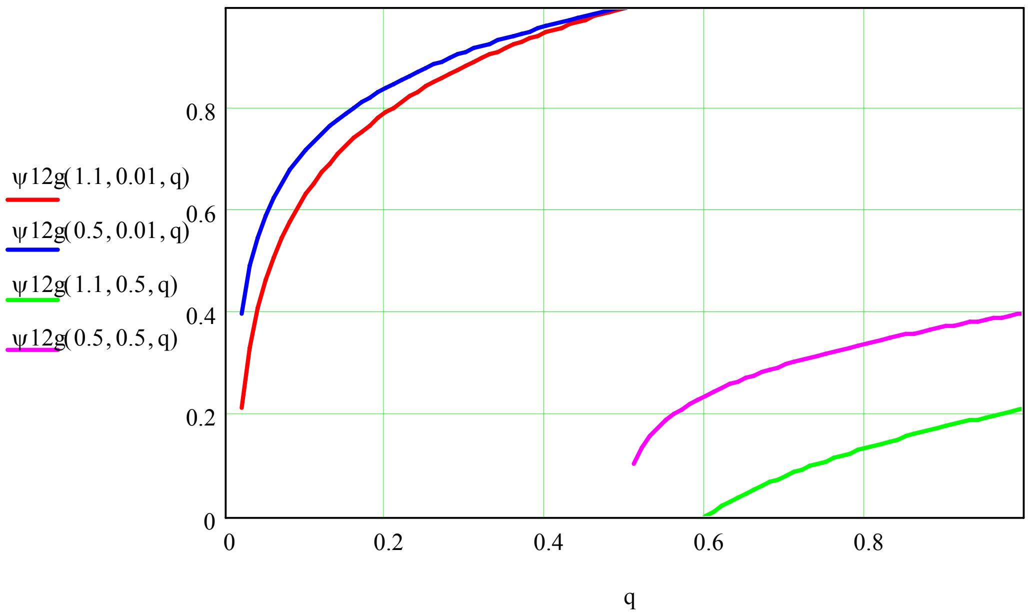


Figure 3. Variation of $\psi 12$ with the probability q for selected coefficients of variation V and fraction η assuming Gumbel distribution.

CHAPTER IV - DESIGN ASSISTED BY TESTING

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Summary

Under particular circumstances it may be favorable or necessary to carry out tests in order to obtain certain design parameters. Typical parameters determined from the tests are actions on the structure, resistance of the structure or structural component and material properties. Tests can be performed also to calibrate parameters in the theoretical model of resistance. The design value of the parameter is obtained from the test results as the estimated value of a certain fractile of the parameter in question. The procedures are explained for the determination of a single property and for the determination of a probabilistic model of resistance.

1 INTRODUCTION

1.1 Background materials

The section 5.2 of the European standard EN 1990 [1] mentions briefly the most general principles of the design assisted by testing and refers to Annex D of the same standard, where the procedures are dealt with in detail. The international standard ISO 2394 [2] also explains the procedures of the design assisted by testing in its Annex D. The two standards differ in a number of details regarding this subject. Some insight in the derivation of the statistical formulations is given in the international standard ISO 12491 [3] and in statistical literature (e.g. [5]).

1.2 General principles

Under particular circumstances it may be favorable or necessary to carry out the tests in order to obtain certain design parameters. The examples of possible such circumstances include:

- calculation models are lacking or are inadequate;
- large number of similar components will be used;
- cases when the calculation model leads to very conservative results;
- derivation of new design formulae;
- confirmation of assumptions made in the design.

The unknown quantities which are evaluated as a result of the tests may be

- actions on the structure (e.g. wind loads);
- structural response under loading or accidental effect;
- strength or stiffness of the structure or structural element.

The level of reliability of a structure designed by testing should be at least the same as for structures designed only by calculation models.

The evaluation of test results should be based on statistical methods. Test results should in principle include probability distribution of unknown quantities, including the statistical uncertainties. This distribution is the base for obtaining the design values and partial factors. The classical statistical interpretation is possible if a large series of tests is performed, or a smaller series of tests is carried out in order to calibrate a model with one or more parameters. When only a small number of tests are performed, no classical statistical interpretation is possible. With the prior information about the distribution of the investigated quantities it is possible to interpret the test results as statistical using Bayesian procedures.

The design values for a material property, a model parameter or resistance should be derived from the tests either by (a) assessing the characteristic value and applying the appropriate partial and conversion factors, or by (b) direct determination of the design value implicitly or explicitly accounting for reliability required and conversion of results.

The derivation of the characteristic value should take into account the scatter of test data, statistical uncertainty associated with the number of tests and prior statistical knowledge. The partial factors should be taken from the appropriate Eurocode. The calculation model should take into account for differences between test arrangement and real behavior.

In case when method (b) is used, the relevant limit states and the required level or reliability should be accounted for.

1.3 Preliminary statistical concepts

The basic idea behind the expressions given for the determination of design values from the tests is the following: the values $x_1,x_2,...,x_n$ of the sample (for example the values obtained in n realizations of the test) can be regarded as n observed values of the same random variable X. However, we may equally well regard these n values as a single observation of n random variables $X_1,X_2,...,X_n$ (a random vector $\boldsymbol{X}$) that have the same distribution (the distribution of X) and are independent, since sample values are assumed to be independent.

Example 1.

When testing n samples we obtained values $x_1,x_2,...,x_n$ for the parameter X. If we assume that the parameter X is random variable with the mean μ and the standard deviation σ, what is the mean m and the standard deviation s of the average value of the sample?

To answer this, we will make use of one of the theorems in probability theory: if we have n independent random variables $X_1,X_2,...,X_n$ and a_i are arbitrary numbers, then the following expressions hold:

$$E(\Sigma(a_i X_i))=\Sigma a_i E(X_i) \tag{1}$$

$$D(\Sigma(a_i X_i))=\Sigma a_i^2 D(X_i) \tag{2}$$

where $E(X)$ and $D(X)$ represent the mean value (expectation) and the variance (dispersion) of the random variable X. The statement about the mean value holds even if the variables X_I are not independent. Note also, that X is distributed arbitrarily.

We can look at the sample $x_1,x_2,...,x_n$ as a realization of the random variables $X_1,X_2,...,X_n$. Then $M=\Sigma(X_i)/n$ is also a random variable whose realization is the average value of the sample. If we put $a_i=1/n$ in the above equations, we get the expressions for the mean m and standard deviation s of the sample average value as follows:

$$m = E(M) = E(\Sigma(X_i)/n) = (\Sigma E(X_i))/n = n\mu /n =\mu \tag{3}$$

$$s^2=D(M) = D(\Sigma(X_i)/n) = (\Sigma D(X_i))/n^2 = n\sigma^2 /n^2 =\sigma^2/n \tag{4}$$

We see that that if the number of samples (tests) increases, the expected value of the mean of the average value stays the same (and equal to the mean of the parameter we are testing), but the standard deviation decreases with the square root of the number of samples. The coefficient of variation of the average value of the sample is

$$V_M = s/m = \sigma/(\mu\sqrt{n}) = V_X/\sqrt{n} \tag{5}$$

and also decreases with the number of samples. The ratio V_M/V_X is shown on the Figure 1.

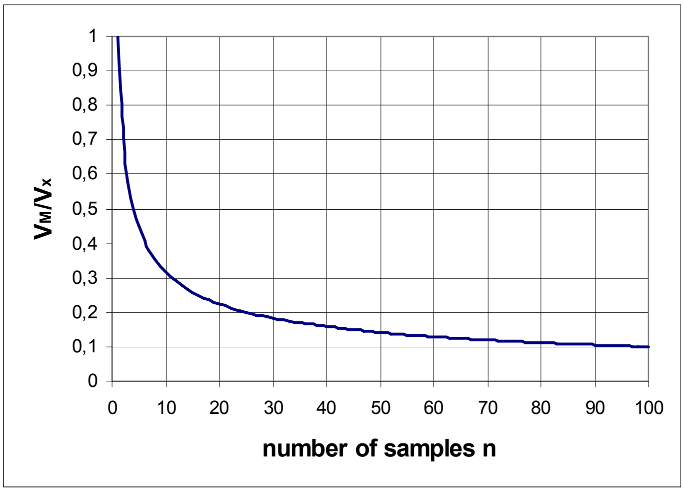


Figure 1. Ratio V_M / V_X as a function of the number of samples. V_M is the coefficient of variation of the average value of the sample and V_X is coefficient of variation of the measured parameter.

As has been already mentioned, no statement has been given about the distribution of the parameter X. The above formulae are valid for arbitrary distribution. If the random variable X is distributed normally, then the sum ΣX_i is also distributed normally and so is the mean M. This follows from the fact that a linear combination of several normal random variables is also a normal random variable (even if these variables are not independent, see e.g. [5]). The similar also holds for a log-normal distribution: if the random variable X is distributed log-normally, then the mean M is also distributed log-normally. This is a direct consequence of the definition: if X is normal random variable, then $Y=\ln(X)$ is log-normal random variable.

Example 2.

In tensile tests the yield stress σ_Y and the tensile strength σ_m of N = 45 specimens of same (steel) material have been measured. Specimens were cylindrical with diameter d. The Table 1 gives the measured values (units are mm and MPa). For each of the random parameter σ_Y , σ_m and d compute the coefficient of variation V_n of the average value of samples n=1,2,…,N and, assuming that $V=V_N$ is the coefficient of variation of the particular parameter, plot a ratio V_n/V as a function of the number of samples n. Compare with the equation (5).

Table 1. The results of the tensile tests.

n	d	σ_Y	σ_m	n	d	σ_Y	σ_m	n	d	σ_Y	σ_m	n	n	σ_Y	σ_m
1	7,98	816	924	13	7,98	832	949	25	8	818	907	37	7,98	829	930
2	8	845	944	14	7,98	811	932	26	7,98	828	940	38	7,98	810	904
3	8	832	948	15	7,97	840	937	27	7,99	817	941	39	7,96	832	925
4	7,99	830	925	16	8	839	934	28	7,97	851	959	40	7,99	823	916
5	7,98	846	969	17	8	855	943	29	7,98	855	970	41	7,98	826	957
6	7,95	821	937	18	7,98	830	928	30	7,99	847	947	42	8	829	931
7	7,98	826	928	19	8	833	934	31	7,98	822	921	43	7,98	815	910
8	7,99	822	934	20	7,98	826	934	32	7,98	836	925	44	7,98	826	942
9	7,96	841	956	21	7,98	836	942	33	7,97	830	945	45	8	823	932
10	7,99	807	946	22	7,98	843	948	34	7,99	830	938				
11	7,97	831	942	23	7,98	840	937	35	7,98	845	945				
12	7,98	830	926	24	8	847	928	36	8	829	934				

First we compute the successive average values $m_1=x_1$, $m_2=(x_1+x_2)/2$, …, $m_N=(x_1+x_2+\ldots+x_N)/N$. Then we treat m_i as a realization of the random variable and we find the mean, standard deviation and the coefficient of variation of successive samples (m_1), (m_1,m_2), …, $(m_1,m_2,\ldots,m_N)$. We repeat this procedure for the three random variables X : the diameter of the specimen, tensile strength and yield stress. The results are shown in Figure 2 as a plot of the coefficient of variation V_m against the number of samples. The difference between analytical and test results is also due to the fact that we don't know the true coefficient of variation V_x. Instead we took a value σ_N/μ_N, computed from the sample mean and sample standard deviation of all the N values for a particular parameter in Table 1.

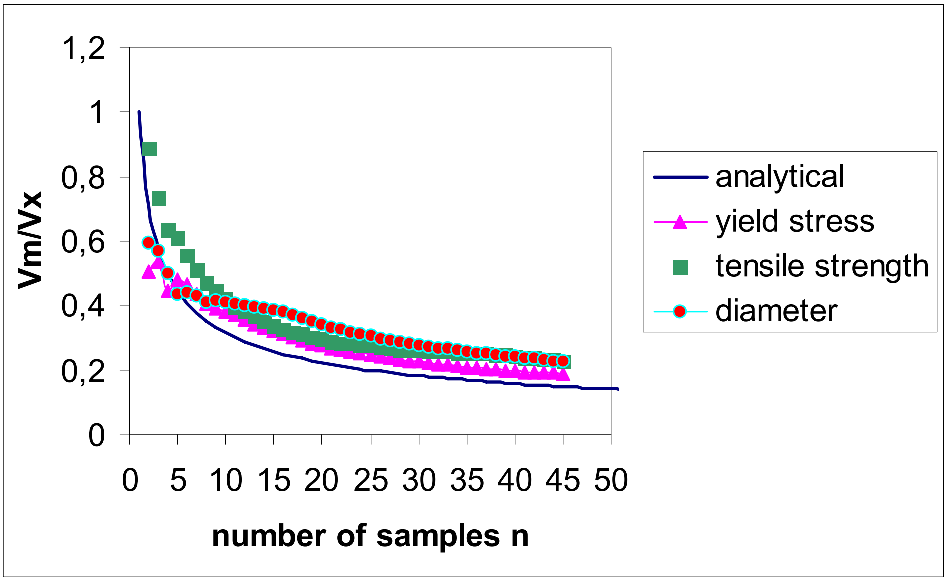


Figure 2. Coefficient of variation V_m of the average value as a function of the number of samples. Comparison of analytical result with results from the tests.

2 STATISTICAL DETERMINATION OF A SINGLE PROPERTY

2.1 General principles

This section gives expressions for deriving design values of the ultimate resistance or serviceability parameters of a structure or a component and for deriving the design values of material properties.

It is assumed that all variables follow normal or lognormal distribution and that there is no prior knowledge about the value of the mean. Two cases are considered regarding the knowledge of the standard deviation, namely "σ_X known" and "σ_X unknown" In EN 1990 [1] the assumptions of the knowledge of σ_X are replaced with assumption of knowledge of the coefficient of variation V_X (see also remarks in the section 4.6 of the Appendix 1 of Handbook H1). In practice it is often preferable to use "V_X unknown" together with a conservative estimate for V_X rather then the assumption "V_X known".

In Eurocode 1990 [1] the determination of characteristic value X_k or design value X_d of the parameter X in question (a material property, resistance or a model) is based on the prediction method of fractile estimation (see section 4.3, Appendix 1 to the Handbook H1 for more details). Similar results are obtained by using the coverage method of fractile estimation with the confidence level 0,75. In the following two examples this method is explained.

Example 3.

In a test we obtained values $x_1, x_2, \ldots, x_n$ for the parameter X, which we assume to be distributed normally. Assuming that the standard deviation σ_x of the population is known, find the value k such that we will have the probability γ (confidence level) that the true mean μ_x of the parameter X will be greater then the sample mean m_x, according to equation:

$$X_k = m_x - k\ \sigma_x \leq \mu_x \tag{6}$$

In other words, we are solving the equation:

$$\mathrm{P}\ (m_x - k\ \sigma_x \leq \mu_x) = \gamma \tag{7}$$

As we have shown (equations (3) and (5)) the sample mean m_x is a random variable with the mean μ_X and the standard deviation $\sigma_X/\sqrt{n}$. If we multiply the above inequality by -1, add a value m_x and divide by $\sigma_X/\sqrt{n}$, we get

$$\mathrm{P}\ ((m_x - \mu_x)\sqrt{n}\ /\sigma_x \leq k\sqrt{n} = u_\gamma) = \gamma \tag{8}$$

where the expression on the left side of the inequality is standardized normal variable and u_γ is a fractile of the standardized normal distribution corresponding to the probability γ . The value k is therefore:

$$k = u_\gamma\ /\sqrt{n} \tag{9}$$

Example 4.

We have the same situation as in the Example 3, but now the standard deviation σ_x is unknown. We are searching k such that we will have the probability γ that the true mean μ_x of the parameter X will be greater then the sample mean m_x, according to equation:

$$X_k = m_x - k\ s_x \leq \mu_x \tag{10}$$

We make use of the following theorem: if all the terms are the same as in Example 3 and s_x is the sample standard deviation:

$$s_x = \Sigma(x_i - m_x)^2/(n\text{-}1) \tag{11}$$

then the random variable

$$(m_x - \mu_x)\sqrt{n}\ /\ s_x \tag{12}$$

has a t-distribution with the degree of freedom n-1.

Now we proceed exactly as in the Example 3 and obtain the value k:

$$k = t_\gamma\ /\sqrt{n} \tag{13}$$

with t_γ being the fractile of the t-distribution of degree n-1 corresponding to the probability γ.

2.2 Assessment via the characteristic value

When determining the design value X_d of the parameter X from the assessed characteristic value X_k , we use the following equation according to Eurocode 1990 [1]:

$$X_d = \eta_d X_k / \gamma_m \tag{14}$$

where the characteristic value X_k is given by

$$X_k = m_x(1 - k_n V_x) \tag{15}$$

This equation is equal to equation (6) with the coefficient of variation V_x given by:

$$V_x = \sigma_x / m_x \tag{16}$$

The γ_m is the partial factor for the parameter X and it should be taken from the appropriate Eurocode EN 1992 to EN 1998. The η_d is the design value of the conversion factor. This factor covers the differences between the laboratory test conditions and conditions during the actual use. The value k_n is obtained from the prediction method of fractile estimation and is:

$$k_n = -u_p (1/n + 1)^{1/2} \tag{17}$$

When the coefficient of variation V_x is known, then u_p is taken as a fractile of the standardized normal distribution corresponding to probability p. For characteristic values of the single property the probability p=0,05 is used, so that u_p = -1.645. When the coefficient of variation V_x is unknown, then a fractile t_p of the t-distribution with the degree of freedom $n-1$ corresponding to probability p is used instead of u_p. The factor k_n is dependent on the number of samples n and is given in Table 2 for two cases, "V_x known" and "V_x unknown" and for probability p=0,05.

Table 2. Values of k_n for the 5% characteristic value.

n	1	2	3	4	5	6	8	10	20	30	∞
V_x known	2,31	2,01	1,89	1,83	1,80	1,77	1,74	1,72	1,68	1,67	1,64
V_x unknown	-	-	3,37	2,63	2,33	2,18	2,00	1,92	1,76	1,73	1,64

The numbers in the Table are actually based on assumptions "σ_x known" and "σ_x unknown". In EN 1990 [1] these assumptions are replaced with assumption of knowledge of the coefficient of variation V_X. If the standard deviation σ_x is known then V_x should be computed from the equation (16). If σ_x or V_x is unknown, then V_x is calculated from the sample standard deviation s_x (equation (11)) as

$$V_x = s_x / m_x \tag{18}$$

The cases above assume that the variable X is distributed normally. What if X is distributed log-normally?

When the parameter X is distributed log-normally, we use the transformation $ln\ X=Y$ to obtain the variable Y which is distributed normally with $N(\mu_Y,\sigma_Y^2)$ (see also Appendix A to the handbook H1). The relationships between the mean and variance of both variables are

$$\mu_Y = \ln(\mu_x^2)/\sqrt{(\sigma_x^2+\mu_x^2)} \tag{19}$$

$$\sigma_Y^2 = \ln(1+\sigma_x^2/\mu_x^2) = \ln(1+V_x^2) \tag{20}$$

If the parameter X is distributed log-normally, then we proceed as follows. We transform all the test results according to the equation

$$y_i = \ln(x_i) \tag{21}$$

and compute the sample mean m_Y from the values y_i :

$$m_Y = \Sigma(y_i)/n \tag{22}$$

Then, if the coefficient of variation V_x is known, we compute σ_Y according to equation (20), V_Y from the equation (16), k_n from Table 2 and then we calculate Y_k from equation (15) using m_Y and V_Y instead of m_x and V_x. If the coefficient of variation V_x is unknown, we compute the sample variance:

$$s_Y^{\ 2} = \Sigma(y_i - m_Y)^2/(n\text{-}1) \tag{23}$$

V_Y from the equation (18), k_n from Table 2 and then we calculate Y_k from equation (15) using m_Y and V_Y instead of m_x and V_x.

Finally we transform the computed characteristic value $Y_k = m_Y\text{-}\ k\ \sigma_Y$ (or $m_Y\text{-}\ k\ s_Y$) of the variable Y to the characteristic value X_k of the original variable X:

$$X_k = \exp(m_Y\text{-}\ k\ \sigma_Y) \tag{24}$$

in case if V_x is known and

$$X_k = \exp(m_Y\text{-}\ k\ s_Y) \tag{25}$$

in case if V_x is unknown. The design value X_d is then calculated using equation (14).

Example 5.

Take the test data from the example 2 and factors γ_m.=1,1 and η_d =0,8. Calculate the design value of the tensile strength via the 5% characteristic value for the first 5 test values. Assume both cases, V_x.=0,05 (known) and V_x.unknown. Assume also both types of distribution: normal and lognormal.

We assume first the normal distribution of the tensile strength. The sample mean of the first n=5 test values is m =942 MPa and the sample standard deviation is s= 18.59 MPa. The coefficient of variation is V_x = 18.59/942=0,01973. For the case "V_x. known" we have from the Table 2 k_n=1,80 and the design value is

$\sigma_{m(d)}$ = 0,8× 942×(1 – 1,80×0,05)/1,1=623,4 MPa.

For the case "V_x. unknown" we have from the Table 2 k_n=2,33 and the design value is

$\sigma_{m(d)}$ = 0,8× 942×(1 – 2,33×0,01973)/1,1=653,6 MPa.

Next we assume that the tensile strength is distributed log-normally. We transform the values of tensile strength using equation (21) and calculate the sample mean m_Y=6,85 ln(MPa) and standard deviation s_Y= 0,01967 ln(MPa). For the case "V_x. known" we have

$V_Y = (\ln(1+0{,}05^2))^{1/2}/6{,}85=0{,}00729$ and s_Y= 0,00729×6,85=0,04997 ln(MPa)

from the Table 2 k_n=1,80, the value

Y_k= 6,85-1,8×0,04997=6,76 ln(MPa)

and the design value is

$\sigma_{m(d)}$ = 0,8× exp(6,76)/1,1= 627,4 MPa.

For the case "V_x. unknown" we have from the Table 2 k_n=2,33, the value

Y_k= 6,85-2,33×0,01967=6,80 ln(MPa)

and the design value is

$\sigma_{m(d)}$ = 0,8× exp(6,80)/1,1=655,7 MPa.

We see that for both types of distribution the design value is greater for the case "V_x. unknown" then for the case "V_x. known". The reason for this is that we have assumed much greater V_x.in the case "V_x. known" that it really is. If we had assumed V_x.=0,015, as obtained from all the test results, then we would get:

Normal distribution, "V_x. known":

$\sigma_{m(d)}$ = 0,8× 942×(1 – 1,80×0,015)/1,1=666,6 MPa.

Log-normal distribution, "V_x. known", $s_Y = (\ln(1+0{,}015^2))^{1/2}$=0,015

Y_k= 6,85-1,8×0,015=6,82 ln(MPa)

$\sigma_{m(d)}$ = 0,8× exp(6,82)/1,1= 668,2 MPa.

2.3 Direct assessment of the design value

When determining the design value X_d directly, we should use the following formula:

$$X_d = \eta_d\, m_x(1 - k_{d,n}\, V_x) \tag{26}$$

In case this method is used, the relevant limit states and the required level or reliability should be accounted for. The conversion factor η_d should cover all uncertainties not covered by the test. The factor $k_{d,n}$ is obtained from the prediction method of fractile estimation with the lower value of about 0,1 % (the probability p = 0,001). When the coefficient of variation V_x is known, then parameter X is assumed to be distributed normally and value $k_{d,n}$ is

$$k_{d,n} = -u_p\,(1/n + 1)^{1/2} \tag{27}$$

with the 0,001 fractile of the standardized normal distribution u_p = -3,09 (the value –3,04 is used in Eurocode 1990 [1]). When the coefficient of variation V_x is unknown, then a fractile t_p of the t-distribution with the degree of freedom n-1 corresponding to probability p=0,001 is used instead of u_p. The factor $k_{d,n}$ is dependent on the number of samples n and is given in Table 3:

Table 3. Values of $k_{d,n}$ for the direct assessment of the design value.

n	1	2	3	4	5	6	8	10	20	30	∞
V_x known	4,36	3,77	3,56	3,44	3,37	3,33	3,27	3,23	3,16	3,13	3,04
V_x unknown	-	-	-	11,40	7,85	6,36	5,07	4,51	3,64	3,44	3,04

Example 6.

With the data from the example 5 calculate the design value of tensile strength using the direct method.

We assume first the normal distribution of the tensile strength. For the case "V_x. known" (V_x.=0,015), we have from the Table 3 $k_{d,n}$=3,37 and the design value is

$\sigma_{m(d)}$ = 0,8× 942×(1 – 3,37×0,015)=715,5 MPa.

For the case "V_x. unknown" we have from the Table 2 k_n=7,85 and the design value is

$\sigma_{m(d)}$ = 0,8× 942×(1 – 7,85×0,01973)=636,9 MPa

When log-normal distribution is assumed, for the case "V_x. known" (V_x.=0,015), we have

Y_k= 6,85-3,37×0,015=6,80 ln(MPa)

$\sigma_{m(d)}$ = 0,8× exp(6,80)= 717,9 MPa.

And for the case "V_x. unknown"

Y_k= 6,85-7,85×0,01967= 6,70 ln(MPa)

$\sigma_{m(d)} = 0{,}8\times \exp(6{,}70)=647{,}1$ MPa.

We obtained higher values with the direct method of calculation. This is because we used the same conversion factor η_d in both cases.

2.4 Approximation of the factors k_n and $k_{d,n}$

Factors k_n and $k_{d,n}$ can be calculated by interpolation from the values in Tables 2 and 3 or, alternatively, by approximation functions:

$k_n = 1{,}655+0{,}672/n$, p=0,05, V_x known
$k_n = n/(-0{,}950+0{,}614\times n)$, p=0,05, V_x unknown
$k_n = 3{,}099+1{,}294/n$, p=0,001, V_x known
$k_n = n/(-0{,}986+0{,}323\times n)$, p=0,001, V_x unknown

Figure 3 shows how these formulas approximate the data from the Tables 2 and 3. The error when using these formulas is typically less then 1%.

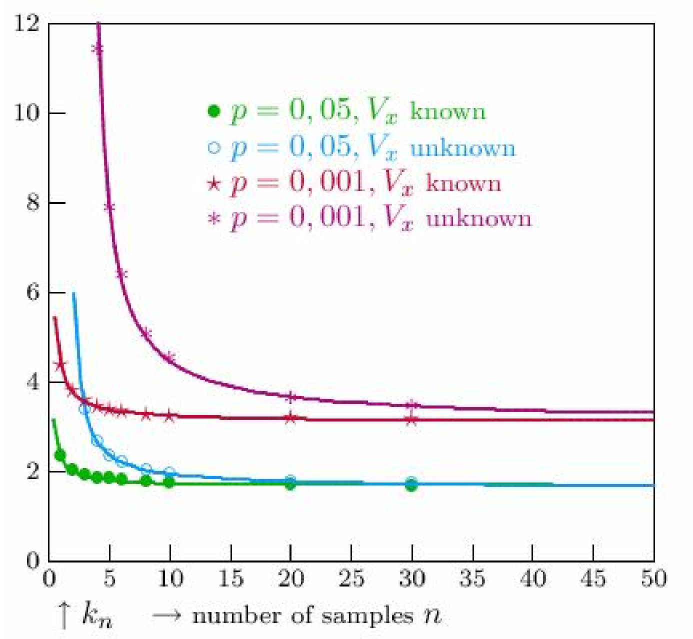


Figure 3. Approximation functions for the factors k_n and $k_{d,n}$.

3 STATISTICAL DETERMINATION OF RESISTANCE MODELS

The procedures given in this section are intended for the calibration of resistance models and for the derivation of design values from the tests undertaken to reduce uncertainties in parameters of the resistance model.

Based on observations and theoretical considerations, a design model of the resistance is developed. The statistical interpretation of the test results should then be used to validate and adjust the model, until sufficient correlation between test and theoretical data is achieved. As in the previous section two methods are considered: (a) by assessing the characteristic value of resistance and (b) by directly assessing the design value of resistance. We start with the method (a).

The following assumptions are made: the resistance function is a function of statistically independent variables $\boldsymbol{X}$=(X_1,..,X_j), which are either normally or log-normally distributed; a sufficient number of tests is carried out; all relevant material and geometrical data are measured.

The first step is to develop a design model for the theoretical resistance r_t

$$r_t = g_{rt}(\boldsymbol{X}) \tag{28}$$

The model should include all relevant basic variables X_i that affect the resistance. We then compare the theoretical model with experimental results. Theoretical values r_{ti} are calculated by substituting actual measured properties of the sample i in the theoretical model. This is to be compared with measured resistance values r_{ei} . We plot the points (r_{ti} , r_{ei}) in a two-dimensional diagram with r_{ti} on the abscissa and r_{ti} on the ordinate, as shown in Figure 4. If the theoretical model is accurate, all the points should lie on the diagonal of the first quadrant. Some scatter will always be present in realistic situations, but if any considerable deviation from that line occurs, further investigation of the experimental procedures and theoretical model should be done.

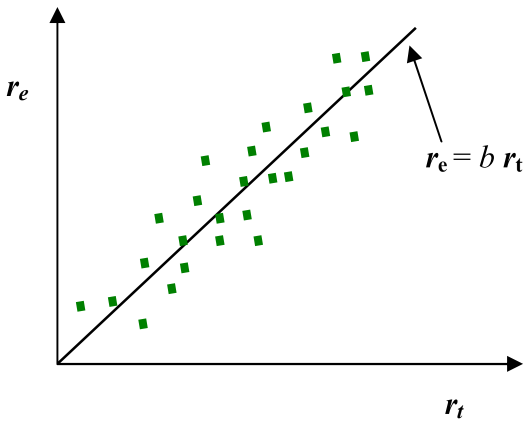


Figure 4. The diagram of the experimental to theoretical resistance.

Next, represent the probabilistic model of the resistance as:

$$r_p = b\ g_{rt}(\mathbf{X})\delta \tag{29}$$

where b is the slope of the least-squares best fit line, given by

$$b = \Sigma(r_e r_t) / \Sigma(r_t^2) \tag{30}$$

δ is the error term, which represents the model uncertainty:

$$\delta = r_e / r_p \tag{31}$$

In absence of other information it is assumed that δ>0 and that it is distributed log-normally. It means that Δ=ln(δ) is distributed normally. For each test value i we calculate

$$\delta_i = r_{ei} / (b r_{ti}) \tag{32}$$

and

$$\Delta_i = \ln(\delta_i) \tag{33}$$

The estimated mean $\overline{\Delta}$ and variance s_Δ for Δ are

$$\overline{\Delta} = \Sigma(\Delta_i)/n \tag{34}$$

$$s_\Delta^2 = \Sigma(\Delta_i - \overline{\Delta})^2 / (n-1) \tag{35}$$

For the estimated value of the coefficient of variation a value

$$V_\delta = \sqrt{\exp(s_\Delta^2) - 1} \tag{36}$$

may be used (derived from the equation (20)).

The coefficients of variation V_{Xi} of the basic variables should now be determined. These can be obtained from test data, if it can be shown that the tests are fully representative of the actual population. Since this is not generally the case, the coefficients of variation V_{Xi} will be established based on prior knowledge or assumptions.

The coefficient of variation V_r of the resistance function is then computed as follows. If the resistance function is a product of the basic variables X_i

$$X = X_1 \times X_2 \times X_3 \times \ldots \times X_\mathrm{j} \tag{37}$$

which are considered to be independent and distributed normally, then by taking logarithm of the above expression

$$\ln(X) = \ln(X_1) + \ln(X_2) + \ln(X_3) + \ldots + \ln(X_\mathrm{j}) \tag{38}$$

we have a sum of log-normally distributed variables. The variance of the sum of independent variables is (see equation (2))

$$\sigma_{ln(X)}{}^2 = \Sigma\ \sigma_{ln(Xi)}{}^2 = \Sigma\ (\ln(1+V_{Xi}{}^2)) \tag{39}$$

where we used the equation (20). Taking the anti-logarithm, we have

$$1+V_{rt}{}^2 = \exp(\sigma_{ln(X)}{}^2) = (1+V_{X1}{}^2)(1+V_{X2}{}^2)\ldots(1+V_{Xj}{}^2) \tag{40}$$

When V_{Xi} are small, the above equation can be simplified to

$$V_{rt}{}^2 = \Sigma(V_{Xi}{}^2) \tag{41}$$

If the resistance function is more complex function, so that it cannot be expressed as a product of basic variables, then V_{rt} is computed from the equation

$$V_{rt}{}^2 = D[g_{rt}(\mathbf{X})]/g_{rt}{}^2(X_m) = \frac{1}{g_{rt}^2(X_m)}\,\Sigma\,\left(\frac{\partial g_{rt}}{\partial X_i}\,\sigma_i\right)^2 \tag{42}$$

where X_m are the mean values of the basic variables.

The coefficient of the variation V_r of the resistance function is then computed as

$$V_r{}^2 = (1+V_\delta{}^2)(1+V_{rt}{}^2) - 1 \tag{43}$$

the standard deviations

$$Q_{rt} = \sigma_{ln(X)} = \sqrt{\ln(1+V_{rt}{}^2)} \tag{44}$$

$$Q_\delta = \sigma_{ln(\delta)} = \sqrt{\ln(1+V_\delta{}^2)} \tag{45}$$

$$Q = \sigma_{ln(r)} = \sqrt{\ln(1+V_r{}^2)} \tag{46}$$

and the weighting factors

$$\alpha_{rt} = Q_{rt}/Q \tag{47}$$

$$\alpha_\delta = Q_\delta/Q \tag{48}$$

The mean value of the resistance function is obtained from the theoretical model using the mean values X_m of the basic variables:

$$r_\mathrm{m} = b\ g_{rt}(X_m) \tag{49}$$

The characteristic resistance r_k is finally obtained from the equation (the derivation of this equation is given in the appendix A):

$$r_k = r_m \exp(-k_\infty\ \alpha_{rt} Q_{rt} - k_n\ \alpha_\delta Q_\delta - 0{,}5\ Q^2) \qquad (50)$$

The factor k_n is taken from the Table 2 for the case "V_x unknown" and k_∞ is the value of k_n for large n (k_∞ =1,64). When only one variable is present in the resistance model, only the term - $k_n\ \alpha_\delta Q_\delta$ from the equation (50) is taken into account.

When the design value of the resistance is assessed directly, rather that from the characteristic value of resistance, then the procedure is the same with the only modification that the values k_∞ and k_n are replaced with the values $k_{\infty,d}$ and $k_{d,n}$, taken from the Table 3 for the case " V_x unknown".

Example 7.

Consider the resistance model in the form $F=A\times\sigma_m$ /1000. As the basic variables we take the section of a rod A and the tensile strength σ_m . Let the mean and coefficient of variation of the basic variables be $E(A)$=269,76 mm^2, $V(A)$=0,00295, $E(\sigma_m)$=936.5 MPa, $V(\sigma_m)$=0,01509. The theoretical $\boldsymbol{F}_t$ and experimental $\boldsymbol{F}_e$ data for various values of the basic variables are given in the Table 4. Compute the characteristic value of the resistance at the mean values of basic variables.

Table 4. Theoretical $\boldsymbol{F}_t$ and experimental $\boldsymbol{F}_e$ data of the resistance in [kN].

n	F_t	F_e	*n*	F_t	F_e	*n*	F_t	F_e
1	214,9	215,9	7	243,6	245,2	13	269,3	270,8
2	219,9	222,4	8	248,1	253,2	14	273,4	274,2
3	224,9	224,7	9	252,5	261,3	15	277,4	279,6
4	229,7	228,5	10	256,8	260,1	16	281,3	281,7
5	234,4	228	11	261	256,5	17	285,2	282,9
6	239,1	239,7	12	265,2	261,7	18	289	278,7

The diagram in Figure 5 shows the data from the Table 4 with the regression line $F_t=b\times F_e$. The value b is obtained from equation (30): b=0.9995.

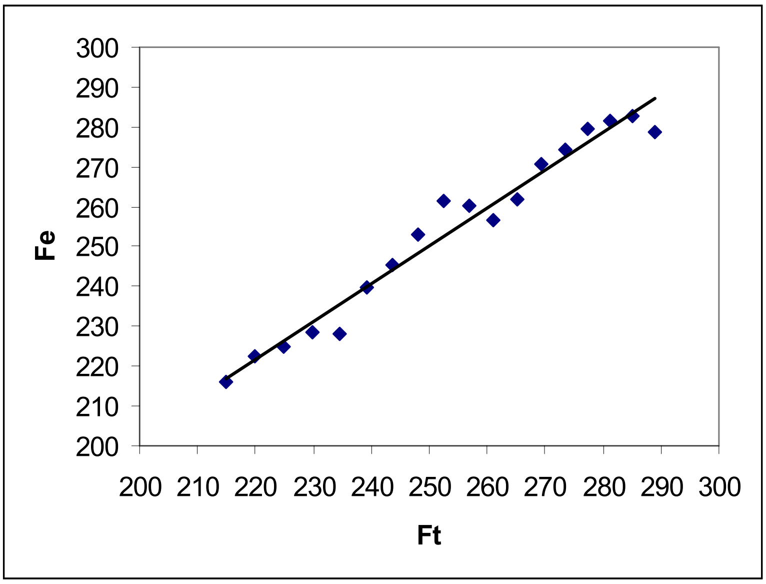


Figure 5. Experimental F_e vs. theoretical F_t resistance.

The mean and sample standard deviation of the logarithm of the error term δ are given by $\overline{\Delta}$=0,000567, s_{Δ}^{2}=0,000277 , the coefficients of variation are calculated from equations (36), (40) and (43) and are V_{δ}^{2}=0,000277, V_{rt}^{2} =0,000236, V_{r}^{2} = 0,000513. The factor k_n=1.78054, is obtained using the approximation formula. The theoretical value of the resistance function at the mean values of basic variables is r_{m}=252,5 kN. Finally, the characteristic value of the resistance is computed:

$$r_{\mathrm{k}}=252,5\times\exp(-1,64\times0,6785\times0,01537-1,7805\times0,7346\times0,0166-0,5\times0.0226^{2})=242.8\text{ kN}$$

We have repeated the above procedure for 10 values of $V(A)$ and $V(\sigma_m)$ equally spaced in the range from 0 to 0,4. The Figure 6 shows the dependence of the characteristic value of the resistance on both coefficients of variation.

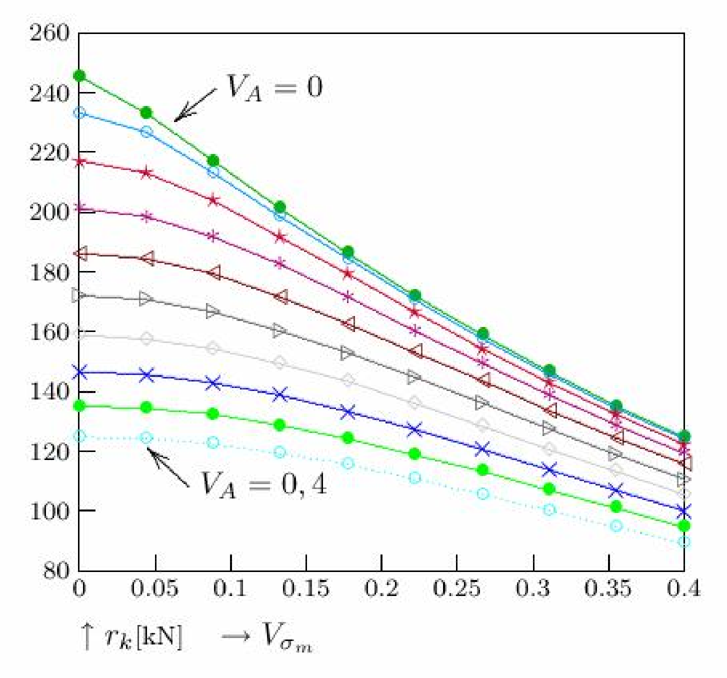


Figure 6. The characteristic value of the resistance for various coefficients of variation of the basic variables A and σ_m .

If the design value is calculated directly, then $k_{\infty,d}$ =3,04 and $k_{n,d}$ = 3,7223 by the approximation formula, and the design value is

$$r_{\mathrm{d}}=252,5\times\exp(-3,04\times0,6785\times0,01537-3,7223\times0,7346\times0,0166-0,5\times0.0226^{2})=233,7\text{ kN}$$

The partial factor γ_{m} for the model resistance in this case is:

$$\gamma_{\mathrm{m}} = r_{\mathrm{m.}} / r_{\mathrm{d}} = 242,8/233,7 = 1,039 \tag{51}$$

Two software products for the calculation of the characteristic resistance, design resistance and partial factors are given in the attachments. These products include the following.

1. The attached Excel workbook **dast.xls** can be used to calculate the values r_k, r_d, and γ_m for different theoretical and experimental data. The user fills in the yellow input fields with the theoretical and experimental values of the resistance, the mean and the coefficient of variation of the basic variables and the mean value of resistance. The worksheet automatically calculates the characteristic and design values of resistance and the partial factor (shown in blue fields) and draws the data in a chart.

2. The computer program **dast.exe** can be used to produce tables of values of these quantities using different values of the indexes of variation of the basic variables. The source file **dast.c** to this computer program is also attached and can be used to produce the **dast.exe** if compiled with a C or C++ compiler.

The program **dast.exe** works in command mode only, i.e. the user must enter the command mode (MSDos or Command Prompt in the Windows system) to use this program. The program reads input data from an input file provided by the user (the program and input files must be at the same, working directory). The format of this input file is as follows. In the first line the word ‘resistance’ is entered. The following lines contain the values of theoretical and experimental values of resistance, two numbers per line. Then a word ‘variations’ is entered, followed by the values of the coefficients of variation V_X for each basic variable, one number per line. If the user requires the values r_k , r_d and γ_m only for the input values of V_X, then the word ‘end’ completes the input file. If the user requires the values r_k , r_d or γ_m for a range of the coefficients of variations V_X , then a word ‘calculate’ is entered followed by the line containing one of the words ‘characteristic’, ‘design’ or ‘partial’, respectfully. Next the word ‘table’ is entered followed with the line containing eight numbers. These numbers indicate the indices of two (arbitrarily chosen) basic variables, the minimum and maximum value of the coefficient of variation for the first basic variable, the minimum and maximum value of the coefficient of variation for the second basic variable, and the number of tabulated values for the first and the second basic variable. The word ‘end’ completes the input. Comment lines starting with ‘;’ are ignored. The input format is also indicated in more detail in the source file **dast.c**.

An example of the use is provided with two input files, **dast.i1** and **dast.i2**, which contain data from the example **7.** The output files **dast.o1** and **dast.o2** that were produced by the command-mode commands **‘dast dast.i1 > dast.o1’** and **‘dast dast.i2 > dast.o2’** are also attached. As shown in the attachment, the second file **dast.o2** can be directly used in Excel to produce three dimensional chart of the calculated quantity in dependence of the indexes of variation of two (arbitrarily chosen) basic variables.

REFERENCES

[1] EN 1990 Eurocode - Basis of structural design. CEN 2002.
[2] ISO 2394 General principles on reliability for structures, ISO 1998.
[3] ISO 12491:1997(E): Statistical methods for quality control of building materials and components, ISO 1997.
[4] JCSS: *Probabilistic model code.* JCSS working materials, http://www.jcss.ethz.ch/, 2000
[5] Kreyszig, E.: Advanced Engineering Mathematics, John Wiley & sons, New York, Chichester, Brisbane, Toronto, Singapore, 1993.

APPENDIX A. The derivation of the equation (50)

Let X be lognormaly distributed. Then the distribution ln(X) is normal with mean $\mu_{\ln X}=\mu(\ln X)$ and standard deviation $\sigma_{\ln X}$. The characteristic value of ln(X) can be written:

$$\ln(X)_{\mathrm{k}} = \mu_{\ln X} - k_n\ \sigma_{\ln X} \tag{A.1}$$

or

$$X_{\mathrm{k}} = \exp(\mu_{\ln X} - k_n\ \sigma_{\ln X}) \tag{A.2}$$

Since the mean $\mu(X)$ of X can be expressed with the mean $\mu_{\ln X}$ and standard deviation $\sigma_{\ln X}$ of ln(X) by the relationship:

$$\mu(X)=\exp(\mu_{\ln X}+\sigma_{\ln X}^2/2) \tag{A.3}$$

from there we have:

$$X_{\mathrm{k}}=\mu(X)\ \exp(-\ k_n\sigma_{\ln X}\ -\ \sigma_{\ln X}^2/2) \tag{A.4}$$

If $X=Y\,Z$ is a product of two influences, Y and Z, then

$$\ln(X) = \ln(Y) + \ln(Z) \tag{A.5}$$

and the mean of ln(X) is:

$$\mu_{\ln X} = \mu(\ln(Y))+\mu(\ln(Z)) \tag{A.6}$$

The standard deviation $\sigma_{\ln X}$ of ln(X) can be expressed using FORM factors α_{Y} and α_{Z} as:

$$\sigma_{\ln X} = \alpha_Y\ \sigma_{\ln Y} + \alpha_Z\ \sigma_{\ln Z} \tag{A.7}$$

If we now combine equations (A.7) and (A.4), then the characteristic value of X as a combination of Y and Z is:

$$X_{\mathrm{k}}=\mu(X)\ \exp(-k_n\ \alpha_Y\ \sigma_{\ln Y} - k_n\ \alpha_Z\ \sigma_{\ln Z} - \sigma_{\ln X}^2/2) \tag{A.8}$$

If we consider the resistance function r as a variable X, the theoretical resistance function r_{t} as variable Y and error term δ as variable Z in the above equation, we rewrite the equation (A.8) with the notation from section 3:

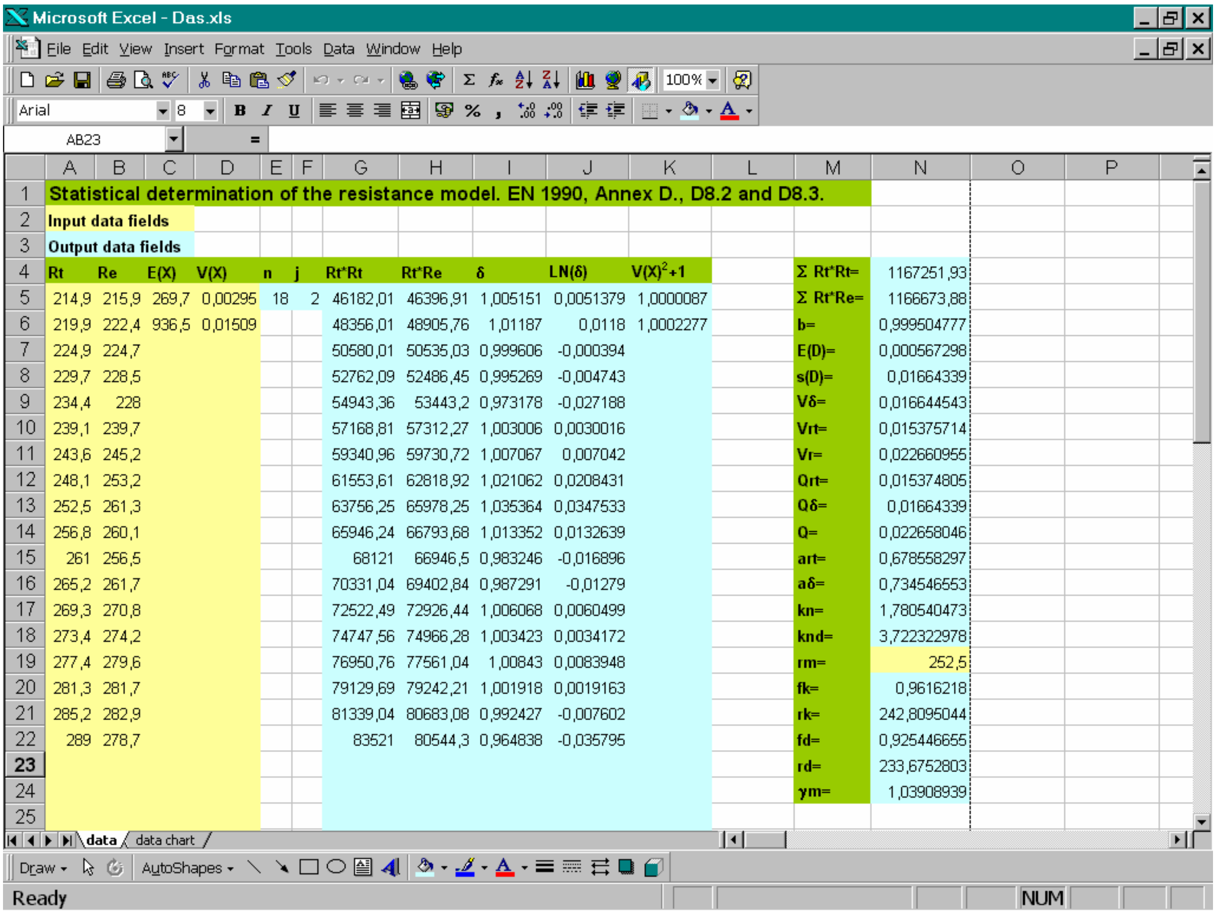
$$r_{\mathrm{k}}= r_{\mathrm{m}}\ \exp(-k_n\ \alpha_{rt}\ Q_{rt} - k_n\ \alpha_\delta\ Q_\delta - Q^2/2) \tag{A.9}$$

Finally, since we assume that there is no statistical uncertainty for theoretical resistance function r_{t} with respect to the number of samples n, we can substitute k_∞ for k_{n} and we obtain the equation (50).

ATTACHMENTS

1. **EXCEL worksheet from the workbook “dast.xls”**
2. **Source file** “dast.c” **to the program** “dast.exe”
3. **Input file “dast.i1” for the program “dast.exe”**
4. **Output file “dast.o1” produced by “dast.exe” from “dast.o1”**
5. **Input file “dast.i2” for the program “dast.exe”**
6. **Output file “dast.o2” produced by “dast.exe” from “dast.o2”**
7. **EXCEL chart showing the data from dast.o2**

Attachment 1 - EXCEL worksheet from the workbook "dast.xls"



	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Statistical determination of the resistance model. EN 1990, Annex D., D8.2 and D8.3.													
2	Input data fields													
3	Output data fields													
4	Rt	Re	E(X)	V(X)	n	j	Rt*Rt	Rt*Re	δ	LN(δ)	V(X)²+1		Σ Rt*Rt=	1167251,93
5	214,9	215,9	269,7	0,00295	18	2	46182,01	46396,91	1,005151	0,0051379	1,0000087		Σ Rt*Re=	1166673,88
6	219,9	222,4	936,5	0,01509			48356,01	48905,76	1,01187	0,0118	1,0002277		b=	0,999504777
7	224,9	224,7					50580,01	50535,03	0,999606	-0,000394			E(D)=	0,000567298
8	229,7	228,5					52762,09	52486,45	0,995269	-0,004743			s(D)=	0,01664339
9	234,4	228					54943,36	53443,2	0,973178	-0,027188			Vδ=	0,016644543
10	239,1	239,7					57168,81	57312,27	1,003006	0,0030016			Vrt=	0,015375714
11	243,6	245,2					59340,96	59730,72	1,007067	0,007042			Vr=	0,022660955
12	248,1	253,2					61553,61	62818,92	1,021062	0,0208431			Qrt=	0,015374805
13	252,5	261,3					63756,25	65978,25	1,035364	0,0347533			Qδ=	0,01664339
14	256,8	260,1					65946,24	66793,68	1,013352	0,0132639			Q=	0,022658046
15	261	256,5					68121	66946,5	0,983246	-0,016896			art=	0,678558297
16	265,2	261,7					70331,04	69402,84	0,987291	-0,01279			aδ=	0,734546553
17	269,3	270,8					72522,49	72926,44	1,006068	0,0060499			kn=	1,780540473
18	273,4	274,2					74747,56	74966,28	1,003423	0,0034172			knd=	3,722322978
19	277,4	279,6					76950,76	77561,04	1,00843	0,0083948			rm=	252,5
20	281,3	281,7					79129,69	79242,21	1,001918	0,0019163			fk=	0,9616218
21	285,2	282,9					81339,04	80683,08	0,992427	-0,007602			rk=	242,8095044
22	289	278,7					83521	80544,3	0,964838	-0,035795			fd=	0,925446655
23													rd=	233,6752803
24													γm=	1,03908939
25														

Attachment 2 - Source file “dast.c” to the program “dast.exe”

```
/*---------------------------------------------------------------------------
 Model resistance calculations according to EN 1990, Anex D, 8.2.2 and 8.2.3.
---------------------------------------------------------------------------*/
#define N_n 100
#define N_j 10
#include <stdio.h>
#include <math.h>

double Rtheor[N_n], Rexper[N_n], Vx[N_j], Va1, Va2, Vb1, Vb2;
int n, nj, na, nb, Xa, Xb, table, calc;
enum { characteristic, design, partial };
char *scalc[]={"characteristic","design","partial"};

/*---------------------------------------------------------------------------
 Model resistance calculations according to EN 1990, Anex D, 8.2.2 and 8.2.3.
 Calculates rk/rm, rd/rm or rk/rd (depending on variable 'calc')
 using D.17a and D.21 in EN 1990, Anex D.

 If table data Xa, Xb, ... are given (see read_data() below), computes a table
 of na*nb values, where the index of variation for the variable Xa is changing
 from Va1 to Va2 and the index of variation for the variable Xb is changing
 from Vb1 to Vb2.
---------------------------------------------------------------------------*/
main(argc,argv) int argc; char **argv; { int i,j; extern double Rmodel();

    if ( !argv[1] ) printf("USAGE: dast inputfile\n"), exit(1);

    read_data(argv[1]);

    if ( table ) {
      printf("%s values for Vx%d=[%g,%g] (|) and Vx%d=[%g,%g] (-->)\n\n",
              scalc[calc],Xa,Va1,Va2,Xb,Vb1,Vb2);
      printf("%7s ","");
      for ( j=0; j<nb; j++ )
        printf("%7.4g ",Vb1+(Vb2-Vb1)/(nb-1)*j);
      for ( i=0; i<na; i++ ) {
        printf("\n%7.4g ",Vx[Xa-1]=Va1+(Va2-Va1)/(na-1)*i);
        for ( j=0; j<nb; j++ ) {
          Vx[Xb-1]=Vb1+(Vb2-Vb1)/(nb-1)*j;
          printf("%7.4g ",Rmodel(n,Rtheor,Rexper,nj,Vx,1,calc));
        }
      }
    }
    else Rmodel(n,Rtheor,Rexper,nj,Vx,0,calc);
}

/*---------------------------------------------------------------------------
 Reads numerical input data:

 Rtheor[]:  theoretical values of resistance model
 Rexper[]:  experimental values of resistance model
 Vx[]:      indexes od variation of basic variables
 Xa, Xb:    indexes of the first and second basic variable to be tabulated
 Va1, Va2:  the range of the index of variation for the first basic variable
 Vb1, Vb2:  the range of the index of variation for the second basic variable
 na, na:    number of tabulated values for the firs and second basic variable

 Data are input through a file in the following format:

 ;comment: the line starting with ';' (or a blank line) is ignored.
 resistance
 Rtheor[1] Rexeper[1]
 Rtheor[2] Rexeper[2]
 ...
 Rtheor[n] Rexeper[n]
```

```
 variations
 Vx[1]
 Vx[2]
 ...
 Vx[j]

 table
 Xa Xb Va1 Va2 Vb1 Vb2 na nb
 calc
 characteristic OR design OR partial
 end
------------------------------------------------------------------------------*/
read_data(s) char *s; { FILE *fd; char buf[501];

   if ( (fd=fopen(s,"r"))==0 ) printf("can't open %s!",s), exit(1);

   for ( ; fgets(buf,500,fd); ) {
     LINE:

     if ( buf[0]==';' || buf[0]=='\n' ) continue;

     else if ( !strncmp("resistance",buf,10) ) {
       for ( ; fgets(buf,500,fd); ) {
         if ( n>=N_n ) printf("too many points!"), exit(1);
         if ( sscanf(buf,"%lg %lg",Rtheor+n,Rexper+n)!=2 ) goto LINE;
         n++;
       }
     }
     else if ( !strncmp("variations",buf,10) ) {
       for ( ; fgets(buf,500,fd); ) {
         if ( nj>=N_j ) printf("too many variables!"), exit(1);
         if ( sscanf(buf,"%lg",Vx+nj)!=1 ) goto LINE;
         nj++;
       }
     }
     else if ( !strncmp("table",buf,5) ) {
       fgets(buf,500,fd);
       if ( sscanf(buf,"%d %d %lg %lg %lg %lg %d %d",
                   &Xa,&Xb,&Va1,&Va2,&Vb1,&Vb2,&na,&nb)!=8 )
                      printf("error in 'table'!"), exit(1);
       table=1;
     }
     else if ( !strncmp("calculate",buf,9) ) {
       fgets(buf,500,fd);
       if      ( !strncmp("design",buf,6) ) calc=design;
       else if ( !strncmp("partial",buf,7) ) calc=partial;
       else if ( !strncmp("characteristic",buf,14) ) calc=characteristic;
       else goto ERROR;
     }
     else if ( !strncmp("end",buf,3) ) break;
     else ERROR: printf("error in file: '%s'",buf), exit(1);
   }
   fclose(fd);

   if ( n<=0 || nj<=0 || nj>n ) printf("error in n/nj!"), exit(1);
   if ( table && (Xa<=0 || Xb<=0 || Xa>nj || Xb>nj) )
     printf("error in table!"), exit(1);
}

/*-----------------------------------------------------------------------------
 Computes the factor of a value of model resistance according to EN 1990,
 Anex D, 8.2.2 and 8.2.3. Depending on the value of calc (characteristic,
 design, partial) returns the exponent from equation D.17a, D.21 and the ratio
 of the two, respectfully.

 If table!=0 prints intermediate and final calculatuions.
------------------------------------------------------------------------------*/
```

```
double Rmodel(n,Rtheor,Rexper,j,Vx,table,calc)
int n,j,table; double Rtheor[],Rexper[],Vx[]; {

  double rt,ret,b,m,s2,t,Vd2,Vrt2,Vr2,Qrt,Qd,Q,art,ad,kn,knd,fd,fk,gm; int i;
  extern double VX2();

  for ( rt=ret=i=0; i<n; i++ ) {
    rt+=  Rtheor[i]*Rtheor[i];
    ret+= Rtheor[i]*Rexper[i];
  }
  b=ret/rt;
  for ( s2=m=i=0; i<n; i++ ) {
    m+= (t=log(Rexper[i]/(b*Rtheor[i])));
    s2+= t*t;
  }
  m/=n;
  s2= n>1? (s2-m*m*n)/(n-1):0.0;
  Vd2=exp(s2)-1;
  Vrt2=VX2(nj,Vx);
  Vr2=(1+Vd2)*(1+Vrt2)-1;
  Qrt=sqrt(log(1+Vrt2));
  Qd=sqrt(log(1+Vd2));
  Q=sqrt(log(1+Vr2));
  art=Qrt/Q;
  ad=Qd/Q;
  kn=n/(-0.95045+0.61443*n);
  knd=n/(-0.98623+0.32344*n);
  fk=exp(-1.64*art*Qrt-kn*ad*Qd-0.5*Q*Q);
  fd=exp(-3.04*art*Qrt-knd*ad*Qd-0.5*Q*Q);
  gm=fk/fd;
  if ( !table ) {
    printf("rt=%g\n",rt);
    printf("ret=%g\n",ret);
    printf("b=%g\n",b);
    printf("m=%g\n",m);
    printf("s=%g\n",sqrt(s2));
    printf("Vd=%g\n",sqrt(Vd2));
    printf("Vrt=%g\n",sqrt(Vrt2));
    printf("Vr=%g\n",sqrt(Vr2));
    printf("Qrt=%g\n",Qrt);
    printf("Qd=%g\n",Qd);
    printf("Q=%g\n",Q);
    printf("art=%g\n",art);
    printf("ad=%g\n",ad);
    printf("kn=%g\n",kn);
    printf("knd=%g\n",knd);
    printf("fk=%g\n",fk);
    printf("fd=%g\n",fd);
    printf("gm=%g\n",gm);
  }
  return calc==partial? gm : (calc==design? fd :fk);
}

/*------------------------------------------------------------------------------
 Computes Vx^2 as a product of (1+Vxi^2)-1.
------------------------------------------------------------------------------*/
double VX2(j,Vx) int j; double Vx[]; { double s; int i;

    for ( s=1, i=0; i<j; i++ ) s*= (Vx[i]*Vx[i]+1);

    return s-1;
}
```

Attachment 3 - Input file “dast.i1” for the program “dast.exe”

```
;Example 7, calculation with fixed indexes of variation
resistance
     214.9    215.9
     219.9    222.4
     224.9    224.7
     229.7    228.5
     234.4      228
     239.1    239.7
     243.6    245.2
     248.1    253.2
     252.5    261.3
     256.8    260.1
       261    256.5
     265.2    261.7
     269.3    270.8
     273.4    274.2
     277.4    279.6
     281.3    281.7
     285.2    282.9
       289    278.7
variations
0.00295
0.01509
end
```

Attachment 4 - Output file “dast.o1” produced by “dast.exe dast.i1 >dast.o1”

```
rt=1167250
ret=1166670
b=0.999505
m=0.000567298
s=0.0166434
Vd=0.0166445
Vrt=0.0153757
Vr=0.022661
Qrt=0.0153748
Qd=0.0166434
Q=0.022658
art=0.678558
ad=0.734547
kn=1.78054
knd=3.72232
fk=0.961622
fd=0.925447
gm=1.03909
```

Attachment 5 - Input file "dast.i2" for the program "dast.exe"

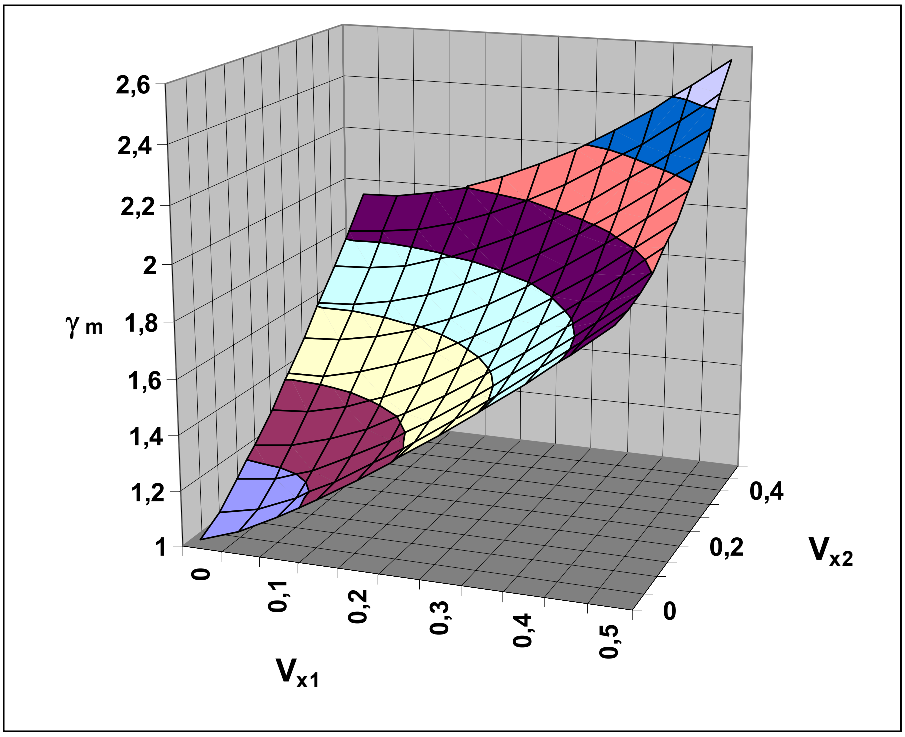
```
;Example 7, calculation of the partial factor (gm) in a table
resistance
     214.9    215.9
     219.9    222.4
     224.9    224.7
     229.7    228.5
     234.4      228
     239.1    239.7
     243.6    245.2
     248.1    253.2
     252.5    261.3
     256.8    260.1
       261    256.5
     265.2    261.7
     269.3    270.8
     273.4    274.2
     277.4    279.6
     281.3    281.7
     285.2    282.9
       289    278.7
variations
0.00295
0.01509
calculate
partial
table
1 2 0 0.5 0 0.5 11 11
```

Attachment 6 - Output file “dast.o2” produced by “dast.exe dast.i2 >dast.o2”

```
partial values for Vx1=[0,0.5] (|) and Vx2=[0,0.5] (-->)
```

	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	1.033	1.08	1.154	1.235	1.322	1.414	1.51	1.611	1.716	1.826	1.939
0.05	1.08	1.109	1.173	1.249	1.333	1.423	1.519	1.619	1.724	1.833	1.946
0.1	1.154	1.173	1.221	1.288	1.366	1.452	1.545	1.644	1.747	1.855	1.967
0.15	1.235	1.249	1.288	1.346	1.417	1.498	1.587	1.683	1.785	1.891	2.002
0.2	1.322	1.333	1.366	1.417	1.482	1.558	1.643	1.736	1.835	1.94	2.05
0.25	1.414	1.423	1.452	1.498	1.558	1.63	1.711	1.801	1.898	2.001	2.109
0.3	1.51	1.519	1.545	1.587	1.643	1.711	1.79	1.877	1.972	2.073	2.18
0.35	1.611	1.619	1.644	1.683	1.736	1.801	1.877	1.962	2.054	2.154	2.26
0.4	1.716	1.724	1.747	1.785	1.835	1.898	1.972	2.054	2.146	2.244	2.349
0.45	1.826	1.833	1.855	1.891	1.94	2.001	2.073	2.154	2.244	2.341	2.446
0.5	1.939	1.946	1.967	2.002	2.05	2.109	2.18	2.26	2.349	2.446	2.549

7. Excel chart from the above data



CHAPTER V - ASSESSMENT OF EXISTING STRUCTURES

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Summary

The approach to assessment of an existing structure is in many aspects different from that taken in designing the structure of a newly proposed building. The effects of the construction process and subsequent life of the structure, during which it may have undergone alteration, deterioration, misuse, and other changes to its as-built (as-designed) state, must be taken into account. To assess existing structures general principles and rules of the Eurocode EN 1990 Basis of structural design must be supplemented by specific procedures provided by International Standards ISO.

1 INTRODUCTION

1.1 Background documents

Background documents related to assessment of existing structures are limited to few national codes and three International Standards ISO 2394 [1], ISO 13822 [2] and ISO 12491 [3]. General principles and rules of the Eurocode EN 1990 Basis of structural design [4] must be supplemented by specific procedures provided in the above mentioned International Standards ISO [1,2,3] that are primarily used in this contribution. Additional information concerning assessment of existing structures may be found in scientific papers and publications, for example in the publications [5], [6] and [7].

1.2 General principles

Assessment of existing structures is becoming a more and more important and frequent engineering task. Continued use of existing structures is of a great significance due to environmental, economic and socio-political assets, growing larger every year. These aspects are particularly relevant to buildings that always constitute a great social and economic value.

General principles of sustainable development regularly lead to the need for extension of the life of a structure, in majority of practical cases in conjunction with severe economic constraints. That is why assessment of existing structures often requires application of sophisticated methods, as a rule beyond the scope of traditional design codes.

The approach to assessment of an existing structure is in many aspects different from that taken in designing the structure of a newly proposed building. The effects of the construction process and subsequent life of the structure, during which it may have undergone alteration, deterioration, misuse, and other changes to its as-built (as-designed) state, must be taken into account. However, even though the existing building may be investigated several times, some uncertainty in behaviour of the basic variables shall always remain. Therefore, similarly as in design of new structures, actual variation in the basic variables describing actions, material properties, geometric data and model uncertainties are taken into account by partial factors or other code provisions.

2 GENERAL FRAMEWORK OF ASSESSMENT

2.1 Reasons for assessment

In general, an existing structure may be subjected to the assessment of its actual reliability in case of:

- rehabilitation of an existing constructed facility during which new structural members are added to the existing load-carrying system;
- adequacy checking in order to establish whether the existing structure can resist loads associated with the anticipated change in use of the facility, operational changes or extension of its design working life;
- repair of an existing structure, which has deteriorated due to time dependent environmental effects or which has suffered damage from accidental actions, for example, earthquake;
- doubts concerning actual reliability of the structure.

In some circumstances assessments may also be required by authorities, insurance companies or owners or may be demanded by a maintenance plan.

2.2 Common rules

Two common rules are usually accepted when assessing existing structures:

- Currently valid codes for verification of structural reliability should be considered, historic codes valid in the period when the structure was designed should be used as guidance documents only.
- Actual characteristics of structural materials, actions, geometric data and structural behaviour should be considered, the original design documentation including drawings should be used as guidance documents only.

The first rule should be applied in order to achieve similar reliability level as in case of newly designed structures. The second principle should avoid negligence of any structural condition that may affect actual reliability (in favourable or unfavourable way) of a given structure.

Most of the current codes are developed assuming the concept of limit states in conjunction with the partial factor method. In accordance with this method, which is mostly considered here, basic variables are specified by characteristic or representative values. The design values of the basic variables are determined on the basis of the characteristic (representative) values and appropriate partial factors.

It follows from the second principle that a visual inspection of the assessed structure should be made whenever possible. Practical experience shows that inspection of the site is also useful to obtain a good feel for actual situation and state of the structure.

As a rule the assessment need not to be performed for those parts of the existing structure that will not be affected by structural changes, rehabilitation, repair, change in use or which are not obviously damaged or are not suspected of having insufficient reliability.

2.3 General procedure

In general, the assessment procedure consists of the following steps (see the flow chart in Annex A to this Chapter):

- specification of the assessment objectives required by the client or authority;
- scenarios related to structural conditions and actions;
- preliminary assessment:
- study of available documentation;
- preliminary inspection;
- preliminary checks;

- decision on immediate actions;
- recommendation for detailed assessment;
- detailed assessment:
- detailed documentary search;
- detailed inspection;
- material testing and determination of actions;
- determination of structural properties;
- structural analysis;
- verification of structural reliability;
- report including proposal for construction intervention;
- repeat the sequence if necessary.

When the preliminary assessment indicates that the structure is reliable for its intended use over the remaining life a detailed assessment may not be required. Conversely if the structure seems to be in dangerous or uncertain condition immediate interventions and detailed assessment may be necessary.

3 INVESTIGATION

3.1 Purpose

Investigation of an existing structure is intended to verify and update the knowledge about the present condition (state) of a structure with respect to a number of aspects. Often, the first impression of the structural condition will be based on visual qualitative investigation. The description of possible damage of the structure may be presented in verbal terms like: 'unknown, none, minor, moderate, severe, destructive'. Very often the decision based on such an observation will be made by experts in purely intuitive way.

A better judgement of the structural condition can be made on the basis of (subsequent) quantitative inspections. Typically, assessment of existing structures is a cyclic process when the first inspection is supplemented by subsequent investigations. The purpose of the subsequent investigations is to obtain a better feel for the actual structural condition (particularly in the case of damage) and to verify information required for determination of the characteristic and representative values of all basic variables. For all inspection techniques, information on the probability of detecting damages if present, and the accuracy of the results should be given.

3.2 Statement

Results of an investigation should be included in the statement that usually contains the data describing

- actual state of the structure;
- types of structural materials and soils;
- observed damages;
- actions including environmental effects;
- available design documentation.

A proof loading is a special type of investigation. Based on such tests one may draw conclusions with respect to:

- the bearing capacity of the tested member under the test load condition;
- other members;
- other load conditions;
- the behaviour of the system.

The inference in the first case is relatively easy; the probability density function of the load bearing capacity is simply cut off at the value of the proof load. The inference from the other conclusions is more complex. Note that the number of proof load tests needs not to be restricted to one. Proof testing may concern one element under various loading conditions and/or a sample of structural elements. In order to avoid an unnecessary damage to the structure due to the proof load, it is recommended to increase the load gradually and to measure the deformations. Measurements may also give a better insight into the behaviour of the system. In general proof loads can hardly address long-term or time-dependent effects. These effects should be compensated by calculation.

4 BASIC VARIABLES

4.1 General

In accordance with the above-mentioned general principles and rules, characteristic and representative values of all basic variables shall be determined taking into account the actual situation and state of the structure. Available design documentation is used as a guidance material only. Actual state of the structure should be verified by its inspection to an adequate extent. If appropriate, destructive or non-destructive inspections should be performed and evaluated using statistical methods.

4.2 Characteristic values

For verification of the structural reliability using partial factor method, the characteristic and representative values of basic variables shall be considered as follows:

(a) Dimensions of the structural elements shall be determined on the basis of adequate measurements. However, when the original design documentation is available and no changes in dimensions have taken place, the nominal dimensions given in the documentation may be used in the analysis.

(b) Load characteristics shall be introduced with the values corresponding with the actual situation verified by destructive or non-destructive inspections. When some loads have been reduced or removed completely, the representative values can be reduced or appropriate partial factors can be adjusted. When overloading has been observed in the past it may be appropriate to increase adequately representative values.

(c) Material properties shall be considered according to the actual state of the structure verified by destructive or non-destructive inspections. When the original design documentation is available and no serious deterioration, design errors or construction errors are suspected, the characteristic values given in original design may be used.

(d) Model uncertainties shall be considered in the same way as in design stage unless previous structural behaviour (especially damage) indicates otherwise. In some cases model factors, coefficients and other design assumptions may be established from measurements on the existing structure (e.g. wind pressure coefficient, effective width values, etc.).

Thus reliability verification of an existing structure should be backed up by inspection of the structure including collection of appropriate data. Evaluation of prior information and its updating using newly obtained measurements is one of the most important steps of the assessment.

5 EVALUATION OF INSPECTION RESULTS

5.1 Updating in general

Using results of an investigation (qualitative inspection, calculations, quantitative inspection, proof loading) the properties and reliability estimates of the structure may be updated. Two different procedures can be distinguished:

(1) Updating of the structural failure probability.

(2) Updating of the probability distributions of basic variables.

Direct updating of the structural reliability (procedure (1)) can be formally carried out using the following basic formula of probability theory:

$$\mathrm{P}(F|I) = \frac{\mathrm{P}(F \cap I)}{\mathrm{P}(I)} \tag{1}$$

where P denotes probability, F local or global failure, I inspection information, and $\cap$ intersection of two events. The inspection information I may consist of the observation that the crack width at the beam B is smaller than at the beam A. An example of probability updating using equation (1) is presented in Annex B to this Chapter.

5.2 Updating of probability distribution

The updating procedure of a univariate or multivariate probability distribution (procedure (2)) is given formally as:

$$\mathrm{f}_X(x|I) = C\,\mathrm{P}(I|x)\,\mathrm{f}_X(x) \tag{2}$$

where $\mathrm{f}_X(x|I)$ denotes the updated probability density function of X, $\mathrm{f}_X(x)$ denotes the probability density function of X before updating, X a basic variable or statistical parameter, I inspection information, C normalising constant, and $\mathrm{P}(I|x)$ likelihood function. An illustration of equation (2) is presented in Figure 1.

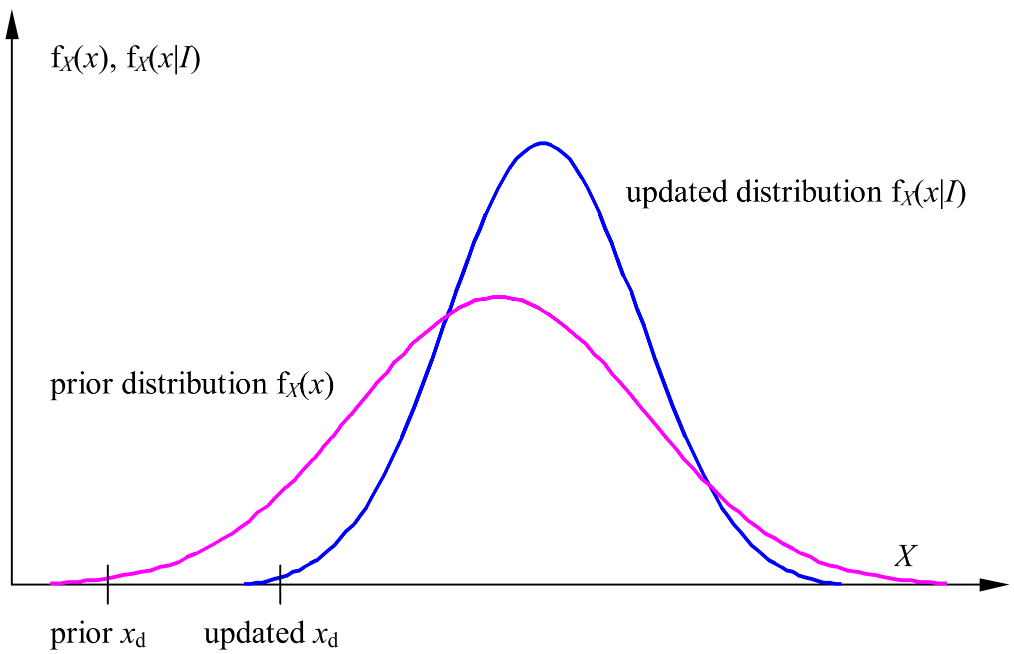

Figure 1. Updating of probability density function for an expected variable X.

In the example shown in Figure 1 updating leads to a more favourable distribution with a greater design value x_d than the prior design value x_d. In general, however, the updated distribution might be also less favourable than the prior distribution.

5.3 Updating of failure probability

Once the updated distributions for the basic variables $f_X(x)$ have been found, the updated failure probability $P(F|I)$ (procedure (1)) may be determined by performing a probabilistic analysis using common method of structural reliability for new structures. Symbolically it can be written

$$P(F|I) = \int_{g(x<0)} f_X(x \mid I)\mathrm{d}x \qquad (3)$$

where $f_X(x|I)$ denotes the updated probability density function and $g(x) < 0$ denotes the failure domain ($g(x)$ being the limit state function). It should be proved that the probability $P(F|I)$, given the design values for its basic variables, does not exceed a specified target value.

5.4 Updating of characteristic and design values

The updating procedure (2) can be used to derive updated characteristic and representative values (fractiles of appropriate distributions) of basic variables to be used in the partial factor method. The Bayesian method for fractile updating is described in Annex C to this Chapter. More information on updating may be found in ISO 12491 [3].

A more practical procedure is to determine directly updated design values for each basic variable. For a resistance parameter X, the design value can be obtained using operational formula of ISO 2394 [1]. For normal and lognormal random variable it holds

$$x_d = \mu(1 - \alpha\beta V) \qquad (4)$$

$$x_d = \mu \exp\left(-\alpha\beta\,\sigma - 0{,}5\sigma^2\right) \qquad (5)$$

where x_d is the updated design value for X, μ updated mean value, α probabilistic influence coefficient, β target reliability index, V updated coefficient of variation, and $\sigma^2 = \ln(1+V^2)$.

The value of the target reliability index β is discussed in ISO/CD 13822 [2], the values of α can be taken equal to those commonly used for new structures (0,7 for the dominating load parameter, 0,8 for the dominating resistance parameter and 0,3 for non-dominating variables according to ISO 2394 [1]).

Alternatively one might determine the characteristic value x_k first and then calculate the design value x_d by applying the appropriate partial factor γ_m:

$$x_d = x_k / \gamma_m \qquad (6)$$

For normal and lognormal random variable X the characteristic value x_k then follows as

$$x_k = \mu(1 - kV) \qquad (7)$$

$$x_k = \mu \exp\left(-k\,\sigma - 0{,}5\sigma^2\right) \qquad (8)$$

where $k = 1{,}64$ (5% fractile of the standardised normal distribution) is usually used. It may be helpful to consider both methods and to use the most conservative result.

This procedure may be applied for all basic variables. However, for geomechanical properties and variable loads other distributions apart from the normal and lognormal distribution may be more suitable.

Note that a lower acceptable reliability level can be specified by reducing β - values for probabilistic design and reducing γ - values in the partial factor method. For a material property X described by a normal distribution the partial factor γ_m may be estimated using equation

$$\gamma_m = \frac{x_k}{x_d} = \frac{\mu - k\sigma}{\mu - \alpha\beta\sigma} \tag{9}$$

which follows from general relationship (6). All the symbols used in (9) are defined above (k = 1,64 is usually used for the characteristic strength). Similar relationships between γ_m and β may be derived for lognormal or other distributions.

6 STRUCTURAL ANALYSIS

Structural behaviour should be analysed using models that describe actual situation and state of an existing structure. Generally the structure should be analysed for ultimate limit states and serviceability limit states using basic variables and taking into account relevant deterioration processes.

All basic variables describing actions, material properties, load and model uncertainties should be considered as mentioned above. The uncertainty associated with the validity and accuracy of the models should be considered during assessment, either by adopting appropriate factors in deterministic verifications or by introducing probabilistic model factors in reliability analysis.

When an existing structure is analysed, conversion factors reflecting the influence of shape and size effect of specimens, temperature, moisture, duration-of-load effect, etc., should be taken into account. The level of knowledge about the condition of components should be also considered. This can be achieved by adjusting the assumed variability in either the load carrying capacity of the components or the dimensions of their cross sections, depending on the type of structure.

When deterioration of an existing structure is observed, the deterioration mechanisms shall be identified and a deterioration model predicting the future performance of the structure shall be determined on the basis of theoretical or experimental investigation, inspection, and experience.

7 VERIFICATION

Reliability verification of an existing structure shall be made using valid codes of practice, as a rule based on the limit state concept. Attention should be paid to both the ultimate and serviceability limit states. Verification may be carried out using partial safety factor or structural reliability methods with consideration of structural system and ductility of components. The reliability assessment shall be made taking into account the remaining working life of a structure, the reference period, and changes in the environment of a structure associated with an anticipated change in use.

The conclusion from the assessment shall withstand a plausibility check. In particular, discrepancies between the results of structural analysis (e.g. insufficient safety) and the real structural condition (e.g. no sign of distress or failure, satisfactory structural performance) must be explained. It should be kept in mind that many engineering models are conservative and cannot be always used directly to explain an actual situation.

The target reliability level used for verification can be taken as the level of reliability implied by acceptance criteria defined in proved and accepted design codes. The target reliability level shall be stated together with clearly defined limit state functions and specific models of the basic variables.

The target reliability level can also be established taking into account the required performance level for the structure, the reference period and possible failure consequences. In accordance with ISO 2394 [1] the performance requirements for assessment of existing structures are the same as for design of a new structure. Lower reliability targets for existing structures may be used if they can be justified on the basis of economical, social and sustainable consideration (see Annex F to ISO 13822 [2]).

An adequate value of the reliability index β should be in general determined [2] considering appropriate reference period. For serviceability and fatigue the reference period equals the remaining working life, while for the ultimate limit states the reference period is in principle the same as the design working life specified for new structures (50 years for buildings). This general approach should be in specific cases supplemented by detailed consideration of the character of serviceability limit states (reversible, irreversible), fatigue (inspectable, not inspectable) and consequences of ultimate limit states (economic consequences, number of endangered people).

8 ASSESSMENT IN THE CASE OF DAMAGE

For an assessment of a damaged structure the following stepwise procedure is recommended:

1) Visual inspection

It is always useful to make an initial visual inspection of the structure to get a feel for its condition. Major defects should be reasonably evident to the experienced eye. In the case of very severe damage, immediate measures (like abandonment of the structure) may be taken.

2) Explanation of observed phenomena

In order to be able to understand the present condition of the structure, one should simulate the damage or the observed behaviour, using a model of the structure and the estimated intensity of various loads or physical/chemical agencies. It is important to have available the documentation with respect to design, analysis and construction. If there is a discrepancy between calculations and observations, it might be worthwhile to look for design errors, errors in construction, etc.

3) Reliability assessment

Given the structure in its present state and given the present information, the reliability of the structure is estimated, either by means of a failure probability or by means of partial factors. Note that the model of the present structure may be different from the original model. If the reliability is sufficient (i.e. better than commonly accepted in design) one might be satisfied and no further action is required.

4) Additional information

If the reliability according to step 3 is insufficient, one may look for additional information from more advanced structural models, additional inspections and measurements or actual load assessment. The updating techniques about how to use this information have been discussed in section 5.

5) Final decision

If the degree of reliability is still too low, one might decide to:
- accept the present situation for economical reasons;
- reduce the load on the structure;
- repair the building;
- start demolition of the structure.

The first decision may be motivated by the fact that the cost for additional reliability is much higher for existing structure than for a new structure. Those who claim that a higher reliability should be generally required for a new structure than for an existing one sometimes use this argument. However, if human safety is involved, economical optimisation has a limited significance.

9 FINAL REPORT AND DECISION

The final report on structural assessment and possible interim reports (if required) should include clear conclusions with regard to the objective of the assessment based on careful reliability assessment and cost of repair or upgrading. The report shall be concise and clear. A recommended report format is indicated in Annex G to ISO 13822 [2].

If the reliability of an existing structure is sufficient, no action is required. If an assessment shows that the reliability of a structure is insufficient, appropriate interventions should be proposed. Temporary intervention may be recommended and proposed by the engineer if required immediately. The engineer should indicate a preferred solution as a logical follow-up to the whole assessment in every case.

It should be noted that the client in collaboration with the relevant authority should make the final decision on possible interventions, based on engineering assessment and recommendations. The engineer performing the assessment might have, however, the legal duty to inform the relevant authority if the client does not respond in a reasonable time.

10 CONCLUDING REMARKS

Assessment of existing structures is usually based on two common rules:

- Currently valid codes for verification of structural reliability are considered, historic codes valid in the period when the structure was designed, should be used only as guidance documents;

- Actual characteristics of structural material, action, geometric data and structural behaviour should be considered; the original design documentation including drawing should be used as guidance material only.

The most important step of the whole assessment procedure is evaluation of inspection data and updating of prior information concerning strength and structural reliability. It appears that a Bayesian approach can provide an effective tool.

Typically, assessment of existing structures is a cyclic process in which the first preliminary assessment is often supplemented by subsequent detailed investigations and assessment. A report on structural assessment prepared by an engineer should include a recommendation on possible intervention. However, the client in collaboration with the relevant authority should make the final decision concerning possible interventions.

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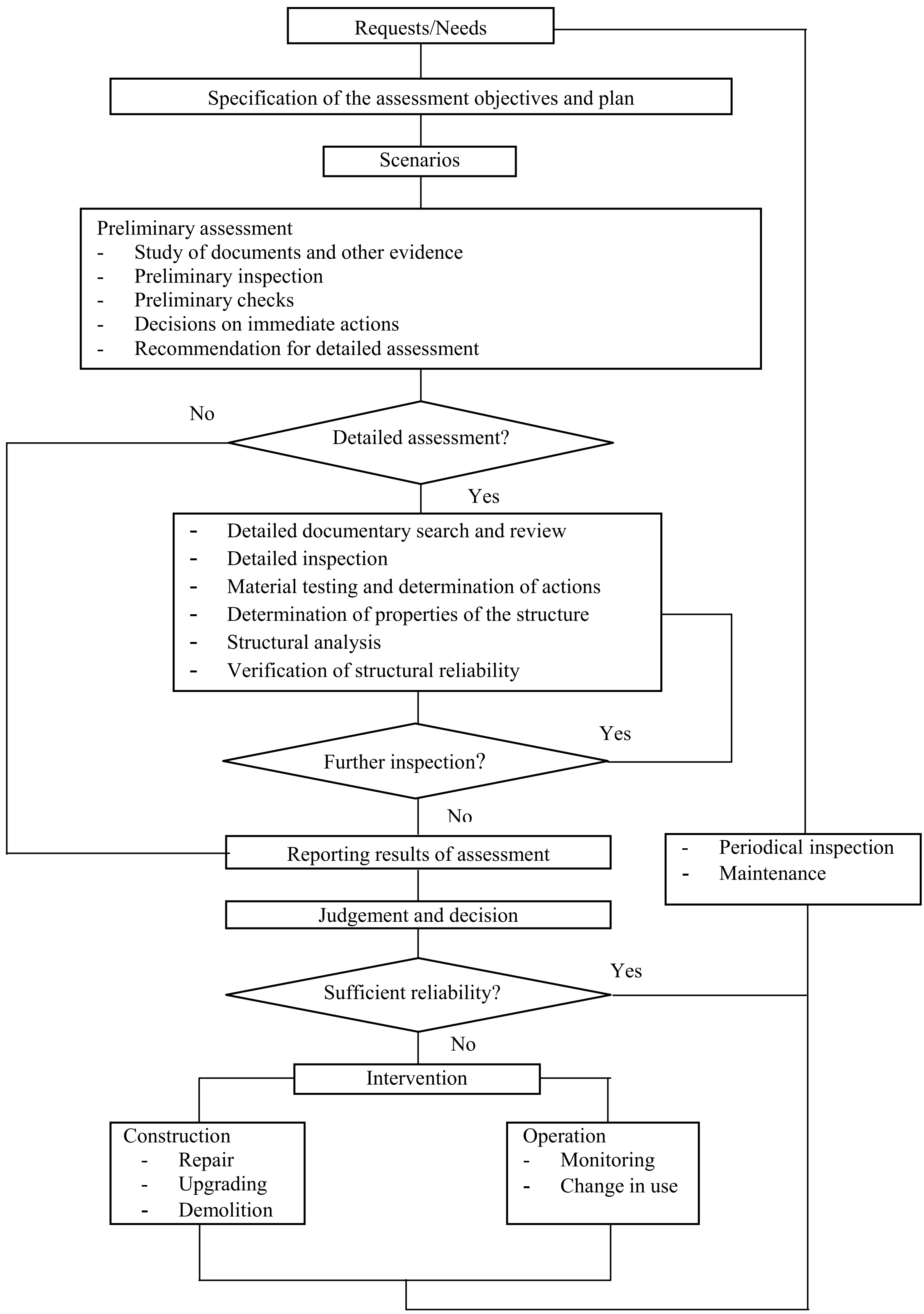
APPENDIX A - GENERAL FLOW OF ASSESSMENT OF EXISTING STRUCTURES



APPENDIX B - PROBABILITY UPDATING

This example of probability updating is adopted from [4] and [5]. Consider the limit state function G(***X***), where ***X*** is a vector of basic variables, and the failure *F* is described by the inequality G(***X***) < 0. If the result of an inspection of the structure *I* is an event described by the inequality $H > 0$ then using equation (1) in the main text the updated probability of failure P(*F*| *I*) may be written as

$$\mathrm{P}(F|\,I) = \mathrm{P}(\mathrm{G}(\boldsymbol{X}) < 0|\,H > 0) = \frac{\mathrm{P}(\mathrm{G}(X) < 0 \cap H > 0)}{P(H > 0)} \tag{B.1}$$

For example consider a simply supported steel beam of the span L exposed to permanent uniform load g and variable load q. The beam has the plastic section modulus W and the steel strength f_y.

Using the partial factor method the design condition $R_d - S_d > 0$ between the design value R_d of the resistance R and design value S_d of the load effect S may be written as

$$W f_{yk} / \gamma_m - (\gamma_g\, g_k L^2/8 + \gamma_q\, q_k L^2/8) > 0 \tag{B.2}$$

where f_{yk} denotes the characteristic strength, g_k the characteristic (nominal) value of permanent load g, q_k the characteristic (nominal) value of variable load q, γ_m partial factor of the steel strength, γ_g the partial factor of permanent load and γ_q the partial factor of variable load. By analogy to (B.2) the limit state function G(***X***) follows as

$$\mathrm{G}(\boldsymbol{X}) = R - S = W f_y - (gL^2/8 + qL^2/8) \tag{B.3}$$

where all the basic variables are generally considered as random variables described by appropriate probabilistic models.

To verify its reliability the beam has been investigated and a proof loading up to the level q_{test} is carried out. It is assumed that g_{act} is the actual value of the permanent load g. If the beam resistance is sufficient the information *I* obtained is described as

$$I = \{H > 0\} = \{W f_y - (g_{act}\, L^2/8 + q_{test}\, L^2/8) > 0\} \tag{B.4}$$

where f_y is the actual steel strength, g_{act} the actual permanent load assuming it has been determined (using non-destructive methods) reasonably accurately.

To determine the updated probability of failure P(*F*| *I*) using equation (B.1) it is necessary to assess the following two probabilities:

$$\mathrm{P}(\mathrm{G}(\boldsymbol{X}) < 0 \cap H > 0) = \mathrm{P}(W f_y - (gL^2/8 + qL^2/8) < 0 \cap W f_y - (g_{act}\, L^2/8 + q_{test}\, L^2/8) > 0) \tag{B.5}$$

$$\mathrm{P}(H > 0) = \mathrm{P}(W f_y - (g_{act}\, L^2/8 + q_{test}\, L^2/8) > 0) \tag{B.6}$$

Additional assumptions concerning the basic variables are needed. Having the results of (B.5) and (B.6) the updated probability of failure P(G(***X***) < 0| $H > 0$) follows from (B.1).

Alternatively, considering results of the proof test, the probability density function $f_R(r)$ of the beam resistance $R = Wf_y$ may be truncated below the proof load as indicated in Figure B.1.

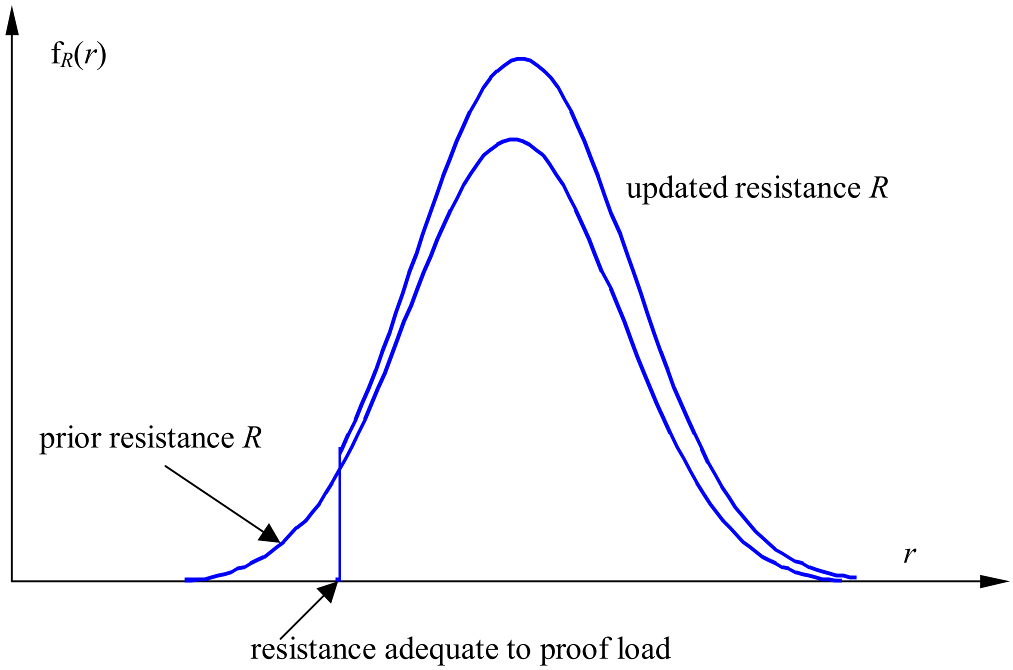


Figure B.1 Truncated effect of proof loading on structural resistance.

Obviously, the truncation of structural resistance R decreases the updated probability of structural failure defined as

$$P_{\mathrm{f}} = \mathrm{P}(R - S < 0) \tag{B.7}$$

and increase, therefore, the updated value of structural reliability.

APPENDIX C - BAYESIAN METHOD FOR FRACTILE ESTIMATION

Fractiles of basic variables can be effectively updated using the Bayesian approach described in ISO 12491 [3]. This procedure is limited here to a normal variable X only for which the prior distribution function $\Pi'(\mu,\sigma)$ of μ and σ is given as

$$\Pi'(\mu,\sigma) = C\,\sigma^{-(1+\nu'+\delta(n'))}\exp\left\{-\frac{1}{2\sigma^2}\left[\nu'(s')^2 + n'(\mu-m')^2\right]\right\} \tag{C.1}$$

where C is the normalising constant, $\delta(n') = 0$ for $n' = 0$ and $\delta(n') = 1$ otherwise. The prior parameters m', s', n', ν' are parameters asymptotically given as

$$\mathrm{E}(\mu) = m',\ \mathrm{E}(\sigma) = s',\ V(\mu) = \frac{s'}{m'\sqrt{n'}}\ ,\ V(\sigma) = \frac{1}{\sqrt{2\nu'}} \tag{C.2}$$

while the parameters n' and ν' are independent and may be chosen arbitrarily (it does not hold that $\nu' = n' - 1$). In equation (C.2) $E(.)$ denotes the expectation and $V(.)$ the coefficient of variation of the variable in brackets. Equations (C.2) may be used to make estimates for unknown parameters n' and ν' provided the values $V(\mu)$ and $V(\sigma)$ are estimated using experimental data or available experience.

The posterior distribution function $\Pi''(\mu,\sigma)$ of μ and σ is of the same type as the prior distribution function, but with parameters m'', s'', n'' and ν'', given as

$$\begin{aligned} n'' &= n' + n \\ \nu'' &= \nu' + \nu + \delta(n') \\ m''n'' &= n'm' + nm \\ \nu''(s'')^2 + n''(m'')^2 &= \nu'(s')^2 + n'(m')^2 + \nu s^2 + nm^2 \end{aligned} \tag{C.3}$$

where m and s are the sample mean and standard deviation, n is the size of the observed sample and $\nu = n-1$. The predictive value $x_{p,\mathrm{pred}}$ of a fractile x_p is then

$$x_{p,\mathrm{Bayes}} = m'' + t_p s''\sqrt{1+1/n''} \tag{C.4}$$

where t_p is the fractile of the t-distribution (see Table C.1) with ν'' degrees of freedom. If no prior information is available, then $n' = \nu' = 0$ and the characteristics m'', n'', s'', ν'' equal the sample characteristics m, n, s, ν. Then equation (C.4) formally reduces to so called prediction estimates of the fractile given as

$$x_{p,\mathrm{pred}} = m + t_p s\sqrt{1+1/n} \tag{C.5}$$

where t_p denotes again the fractile of the t-distribution (Table C.1) with ν degrees of freedom. Furthermore, if the standard deviation σ is known (from the past experience), then $\nu = \infty$ and s shall be replaced by σ.

Example

A sample of $n = 5$ concrete strength measurements having the mean $m = 29{,}2$ MPa and standard deviation $s = 4{,}6$ MPa is to be used to assess the characteristic value of the concrete strength $f_{\mathrm{ck}} = x_p$, where $p = 0{,}05$. If no prior information is available, then $n' = \nu' = 0$ and the characteristics m'', n'', s'', ν'' equal the sample characteristics m, n, s, ν. The predictive value of x_p then follows from (C.5) as

$$f_{cu,Bayes} = 23{,}9 - 1{,}8 \times 4{,}3 \times \sqrt{\frac{1}{6} + 1} = 15{,}5\ \text{MPa}$$

where the value $t_p = -2.13$ is taken from Table C.1 for $1 - p = 0.95$ and $\nu = 5 - 1 = 4$. When information from previous production is available the Bayesian approach can be effectively used. Assume the following prior information

$$m' = 30{,}1\ \text{MPa},\ V(m') = 0{,}50,\ s' = 4{,}4\ \text{MPa},\ V(s') = 0{,}28$$

It follows from equation (C.2)

$$n' = \left(\frac{4{,}6}{30{,}1}\frac{1}{0{,}50}\right)^2 < 1\ ,\ \nu' = \frac{1}{2}\frac{1}{0{,}28^2} \approx 6$$

The following characteristics are therefore considered: $n' = 0$ and $\nu' = 6$. Taking into account that $\nu = n - 1 = 4$, equations (C.3) yield

$$n'' = 6,\ \nu' = 11,\ \overline{x''} = 23{,}9\ \text{MPa},\ s'' = 4{,}3\ \text{Mpa}$$

and finally it follows from equation (C.4)

$$f_{cu,Bayes} = 23{,}9 - 1{,}8 \times 4{,}3 \times \sqrt{\frac{1}{6} + 1} = 15{,}5\ \text{MPa}$$

where the value $t_p = -1.81$ is taken from Table C.1 for $1 - p = 0.95$ and $\nu = 10$.

In this example the resulting characteristic strength is greater (by about 10 %) than the value obtained by prediction method without using prior information. Thus, when previous information is available the Bayesian approach may improve (not always) the fractile estimate, particularly in the case of a great variance of the variable. In any case, however, due caution should be paid to the origin of the prior information with regard to the nature of considered variable.

Table C.1 - Fractiles $-t_p$ of the t-distribution with ν degrees of freedom

ν	$1-p$					ν	$1-p$				
	0,90	0,95	0,975	0,99	0,995		0,90	0,95	0,975	0,99	0,995
3	1,64	2,35	3,18	4,54	5,84	12	1,36	1,78	2,18	2,68	3,06
4	1,53	2,13	2,78	3,75	4,60	14	1,35	1,76	2,14	2,62	2,98
5	1,48	2,02	2,57	3,37	4,03	16	1,34	1,75	2,12	2,58	2,92
6	1,44	1,94	2,45	3,14	3,71	18	1,33	1,73	2,10	2,55	2,88
7	1,42	1,89	2,36	3,00	3,50	20	1,32	1,72	2,09	2,53	2,85
8	1,40	1,86	2,31	2,90	3,36	25	1,32	1,71	2,06	2,49	2,79
9	1,38	1,83	2,26	2,82	3,25	30	1,31	1,70	2,04	2,46	2,75
10	1,37	1,81	2,23	2,76	3,17	∞	1,28	1,64	1,96	2,33	2,58

ATTACHMENTS

1. MATHCAD sheet "Update.mcd"

MATHCAD sheet Update is intended for determination of updated probability using Bayes formula. Prior probabilities and likelihoods are taken from file "update.prn"

2. MATHCAD sheet "BayesFract.mcd"

MATHCAD sheet BayesFract intended for determination of the characteristic and design values and material partial factor γM using test data in accordance to EN 1990, Annex D.

Attachment 1 - MATHCAD sheet “Update.mcd“

"Update.mcd" is MATHCAD sheet for probability updating

Probability updating using Bayes formula

$$P(B_i \mid A) = \frac{P(B_i)P(A \mid B_i)}{\sum_{j=1}^{n} P(B_j)P(A \mid B_j)} \qquad \text{or} \qquad p_i'' = \frac{p_i' l_i}{\sum_j p_j' l_j}$$

where p' ~ $P(B_i)$ denotes prior probabilities, l~ $P(A|B_i)$ likelihoods and p"~ $P(B_i)$ updated (posterior) probabilities.

1 Reading data for apriory probabilties and likelihoods from file

$DATA := READPRN("Update.prn")$ Check vaue

Apriory probabilities $p := DATA^{\langle 0 \rangle}$ $n := \text{length}(p)$ $n = 2$

Likelihoods $l := DATA^{\langle 1 \rangle}$

2 Updated (posterior) probabilities p" $_i$

$$pp := \frac{\overrightarrow{(p \cdot l)}}{p \cdot l} \qquad pp = \begin{pmatrix} 0.889 \\ 0.111 \end{pmatrix}$$

3 Alternative specification of input data using directly this sheet

Prior probabilities p_i' $p := (0.8\ \ 0.2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T$

Likelihoods l_i $l := (1.\ \ 0.5\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)^T$

4 Updated (posterior) probabilities p" $_i$

$$pp := \frac{\overrightarrow{(p \cdot l)}}{p \cdot l}$$

5 Listing of the updated probabilities

pp =

	0
0	0.889
1	0.111
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0

$pp_0 = 0.889$

$pp_1 = 0.111$

Attachment 2 - MATHCAD sheet “BayesFract.mcd”

MATHCAD sheet "BayesFract" for estimation of sample fractile

MATHCAD sheet for determination of the characteristic and design values and material partial factor γM using test data in accordance to EN 1990, Annex D.

1. Coefficients of variations of prior mean and stadard deviation

$V\sigma := .01, 0.02..\ 0.8 \qquad V\mu := .01, 0.02..\ 0.8$

2. Prior and current observations

Prior obcervations: $m1 := 30.1 \qquad s1 := 4.4$

Current observation: $m := 29.2 \qquad s := 4.6 \qquad n := 5 \qquad \nu := n - 1$

3. Estimates of prior n' and ν' assuming V_μ and V_σ

Size n1 $\quad n1(V\mu) := \mathrm{floor}\left[\left(\frac{s1}{m1 \cdot V\mu}\right)^2\right] \qquad n1(0.5) = 0$

Degrees of freedom ν1 $\quad \nu 1(V\sigma) := \mathrm{floor}\left(\frac{1}{2 \cdot V\sigma^2}\right) \qquad \nu 1(0.28) = 6$

Updated size n2 $\quad n2(V\mu) := n + n1(V\mu) \qquad n2(0.5) = 5$

Updated ν2 $\quad \nu 2(V\mu, V\sigma) := \left| \begin{array}{ll} \nu + \nu 1(V\sigma) - 1 & \text{if } n1(V\mu) \ge 1 \\ \nu + \nu 1(V\sigma) & \text{otherwise} \end{array} \right. \qquad \nu 2(0.5, 0.28) = 10$

4. Updated means and standard deviations

$$m2(V\mu) := \frac{m \cdot n + m1 \cdot n1(V\mu)}{n2(V\mu)} \qquad m2(0.5) = 29.2$$

$$s2(V\mu, V\sigma) := \sqrt{\frac{\nu \cdot s^2 + \nu 1(V\sigma) \cdot s1^2 + n \cdot m^2 + n1(V\mu) \cdot m1^2 - n2(V\mu) \cdot m2(V\mu)^2}{\nu 2(V\mu, V\sigma)}} \qquad s2(0.5, 0.3) = 4.49$$

$$s''^2 = (\nu s^2 + \nu' s'^2 + n\, m^2 + n' m'^2 - n'' m''^2) / \nu''$$

5. Coeficients of fractile estimates for probability

select the probability $\quad p := 0.05$

p fractile V unknown
qt inverse Student's
distribution

$$ks(V\mu, V\sigma) := qt(p, \nu 2(V\mu, V\sigma)) \sqrt{1 + \frac{1}{n2(V\mu)}}$$

p fractile V known
qnorm is inverse
normal distribution

$$k\sigma(V\mu) := qnorm(p, 0, 1) \cdot \sqrt{1 + \frac{1}{n2(V\mu)}}$$

6. Fractile estimates

Standard deviation s2 unknown

$$xp(V\mu, V\sigma) := m2(V\mu) + ks(V\mu, V\sigma)\, s2(V\mu, V\sigma) \qquad xp(0.5, 0.28) = 20.303$$

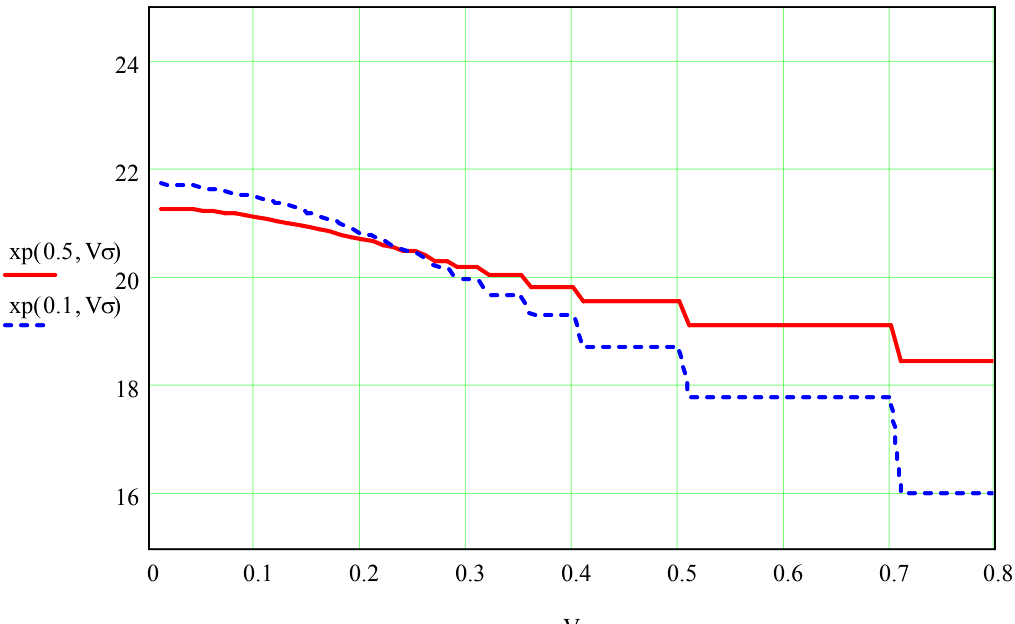


Figure 1. Variation of the frictile xp with V_σ for selected V_μ.

Standard deviation s2 is known, for example s2=s2(0.5,0.28)

$$x\sigma p(V\mu) := m2(V\mu) + k\sigma(V\mu)\, s2(0.5, 0.28) \qquad x\sigma p(0.5) = 21.126$$

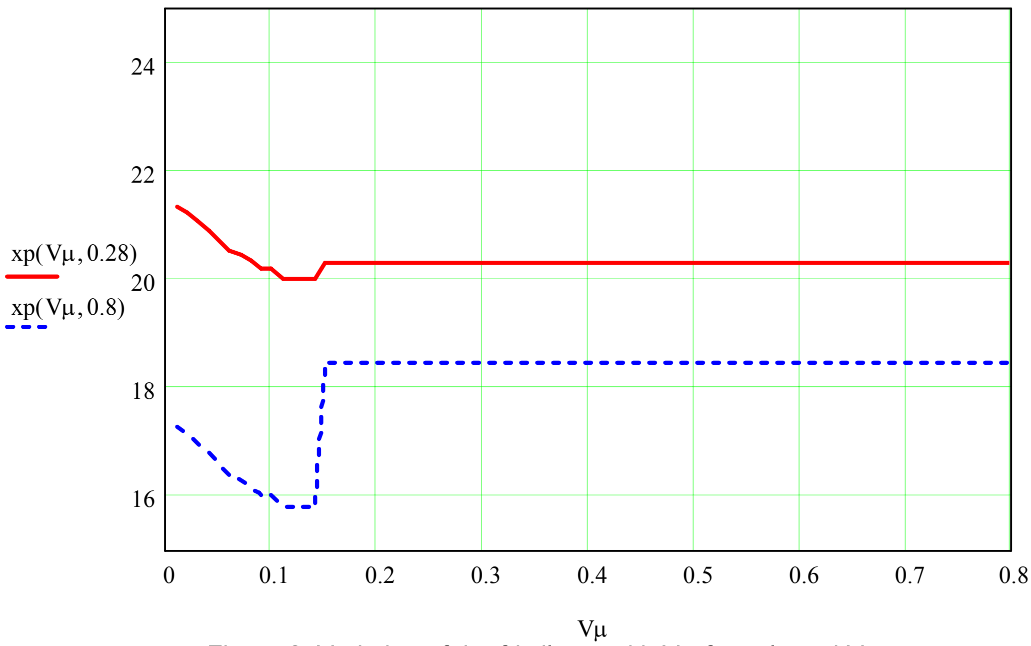
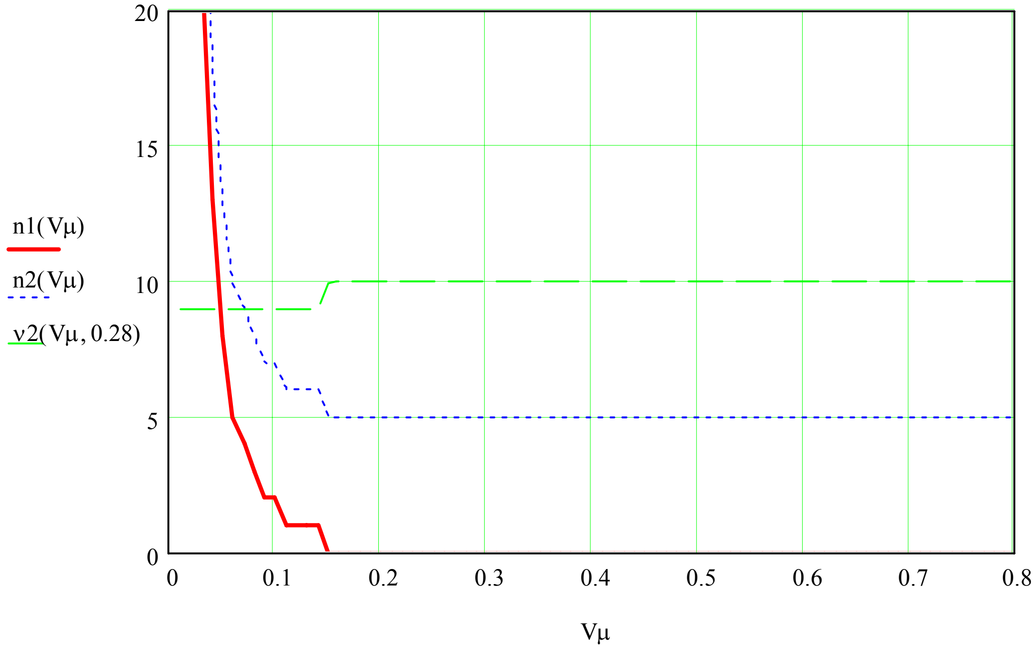


Figure 2. Variation of the frictile xp with V_μ for selected V_σ.

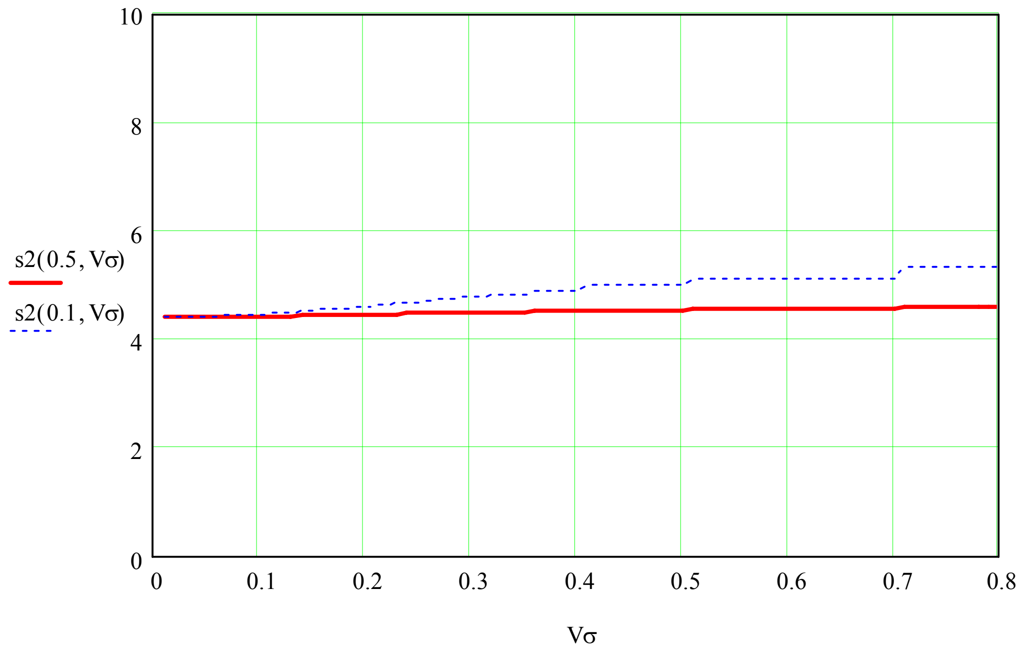
n1(0.5) = 0

n1(0.1) = 2



m2(0.5) = 29.2

m2(0.1) = 29.457



CHAPTER VI - PRINCIPLES OF RISK ASSESSMENT

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Summary

Traditional methods for designing of civil engineering structures and other engineering systems are frequently based on the concept of target probability of failure. However, this fundamental quantity is usually specified on the basis of comparative studies and past experience only. Moreover, probabilistic design methods suffer from several deficiencies, including lack of consideration for accidental and other hazard situations and their consequences. Both of these extreme conditions are more and more frequently becoming causes of serious failures and other adverse events. Available experience clearly indicates that probabilistic design procedures may be efficiently supplemented by a risk analysis and assessment, which can take into account various consequences of unfavourable events. It is therefore anticipated that in addition to traditional probabilistic concepts the methods of advanced engineering design will also commonly include criteria for acceptable risks.

1 INTRODUCTION

1.1 Background documents

Background documents of the risk analysis of civil engineering systems considered in this contribution consist of a number of national and international documents [1] to [9]. It should be noted that Eurocode EN 1990 [10] for design of civil structures is based on the concept of the target probability of failure p_{d}. However, it is well recognised that the reliability of structures and other engineering systems suffers from a number of uncertainties that can hardly be analysed and well described by probabilistic methods [11, 12]. Moreover, traditional probabilistic concepts consider the significance of failure and other adverse events only very vaguely [10]. That is why probabilistic methods are often supplemented by recently developing methods of risk assessment [12]. In some countries, risk assessment even becomes compulsory by law in the case of complex technical systems (power stations, tunnel routes).

1.2 General principles

General principles of the risk analysis and the common tools applied for investigating civil engineering systems considered in this contribution follow the basic concepts presented in documents [1] to [9]. The risk analysis is an important part of the risk assessment and the entire risk management of a system as indicated in Figure 1 (adopted from [2]).

The risk analysis of a system consists of the use of all available information to estimate the risk to individuals or populations, property or the environment, from identified hazards. Risk assessment further includes risk evaluation (acceptance or treatment) as indicated in Figure 1 (adopted from [2]). The whole procedure of the risk assessment is typically an iterative process as indicated in Figure 2 (adopted from [9]). The first step in the risk analysis involves the context (scope) definition related to the system and the subsequent identification of hazards.

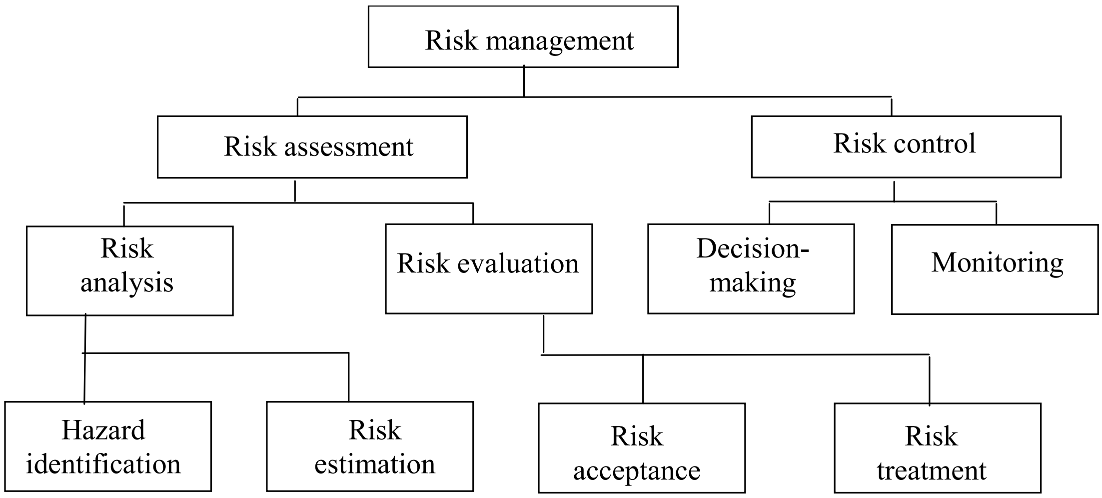

Figure 1. A framework for risk management (adopted from [2]).

The system is understood [2] as a bounded group of interrelated, interdependent or interacting elements forming an entity that achieves in its environment a defined objective through interaction of its parts. In the case of technological hazards related to civil engineering works, a system is normally formed from a physical subsystem, a human subsystem, their management, and the environment. Note that the risk analysis of civil engineering systems (similarly as analysis of most systems) involves usually several interdependent components (e.g. human life, injuries, economic loss).

Any technical system may be exposed to a multitude of possible hazard situations. In the case of civil engineering facilities, hazard situations may include both, environmental effects (wind, temperature, snow, avalanches, rock falls, ground effects, water and ground water, chemical or physical attacks, etc.) and human activities (usage, chemical or physical attacks, fire, explosion, etc.). As a rule hazard situations due to human errors are more significant than hazards due to environmental effects.

2 HAZARD IDENTIFICATION

A hazard is a set of circumstances, possibly occurring within a given system, with the potential for causing events with undesirable consequences. For instance the hazard of a civil engineering system may be a set of circumstances with the potential to an abnormal action (e.g. fire, explosion) or environmental influence (flooding, tornado) and/or insufficient strength or resistance or excessive deviation from intended dimensions. In the case of a chemical substance, the hazard may be a set of circumstances likely to cause its exposure [2].

Hazard identification and modelling is a process to recognize the hazard and to define its characteristics in time and space. In the case of civil engineering systems the hazards H_i may be linked to various design situations of the building (as defined in [7]) including persistent, transient and accidental design situation. As a rule H_i are mutually exclusive situations (e.g. persistent and accidental design situations of a building). Then if the situation H_i occurs with the probability $P\{H_i\}$, it holds $\sum P\{H_i\} = 1$. If the situations H_i are not mutually exclusive, then the analysis becomes more complicated.

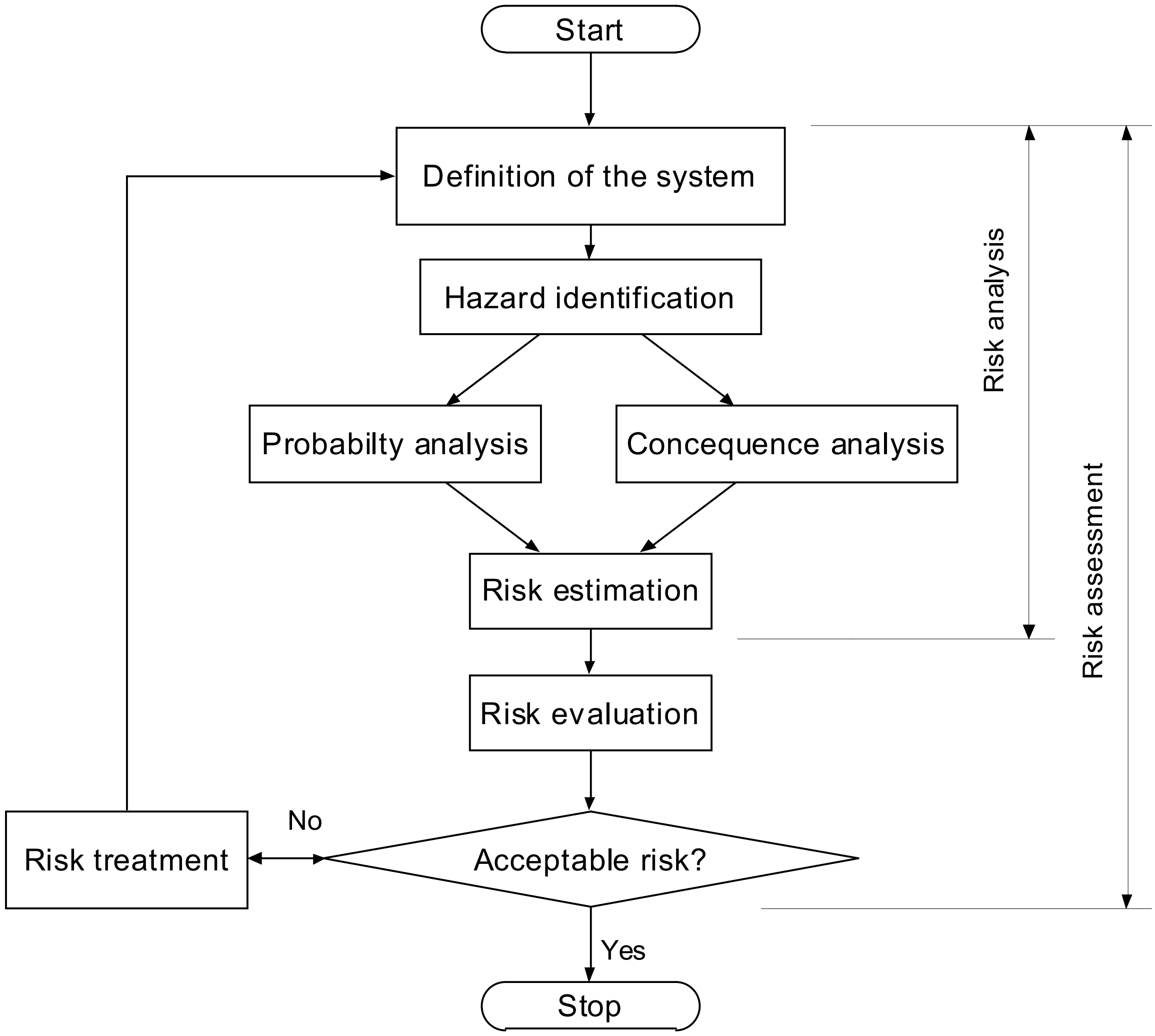

Figure 2. Flowchart of iterative procedure for the risk assessment (adopted from [9]).

Note that in some documents (for example in the recent European document EN 1990 [10]) the hazard is defined as an event, while in risk analysis [2] it is usually considered as a condition with the potential for causing event, thus as a synonym to danger.

3 DEFINITION AND MODELLING OF RELEVANT SCENARIOS

Hazard scenario is a sequence of possible events for a given hazard leading to undesired consequences. To identify what might go wrong with the system or its subsystem is the crucial task to risk analysis. It requires detail examination and understanding of the system [6]. Nevertheless, a given system is often a part of a larger system. Consequently, modelling and subsequent analysis of the system is a conditional analysis.

The modelling of relevant scenarios may be dependent on specific characteristics of the system. For this reason a variety of techniques have been developed for the identification of hazards (e.g. PHA HAZOP) and for the modelling of relevant scenarios (fault tree, event tree/decision trees, causal networks). Detail description of these techniques is beyond the scope of this contribution, may be however found in [6, 9] and other literature.

4 ESTIMATION OF PROBABILITIES

Probability is generally the likelihood or degree of certainty of a particular event occurring during a specified period of time. In particular, reliability of a structure is often expressed as probability related to a specific requirement and a given period of time, for example 50 years [3,10].

Assuming that a system may be found in mutually exclusive situations H_i, and the failure F of the system (e.g. of the structure or its element) given a particular situation H_i occurs with the conditional probability $\mathrm{P}\{F|H_i\}$, then the total probability of failure p_F is given by the law of total probability (see for example [11]) as:

$$p_F = \sum_i \mathrm{P}\{H_i\}\mathrm{P}\{F \mid H_i\} \tag{1}$$

Equation (1) can be used for the modification of the partial probabilities $\mathrm{P}\{H_i\}\mathrm{P}\{F|H_i\}$ (appropriate to the situations H_i) with the aim to comply with the design condition $p_F < p_\mathrm{t}$, where p_t is a specified target probability of failure. The target value p_t may be determined using the probabilistic optimisation of an objective function describing, for example, the total cost.

The conditional probabilities $\mathrm{P}\{F|H_i\}$ must be determined by a detail probabilistic analysis of the respective situations H_i under relevant scenarios. The traditional reliability methods [8] assume that the failure F of the system can be well defined in the domain of the vector of basic variables $\boldsymbol{X}$. For example, it is assumed that a system failure may be defined by the inequality $\mathrm{g}(\boldsymbol{x}) < 0$, where $\mathrm{g}(\boldsymbol{x})$ is the so called limit state function, where $\boldsymbol{x}$ is a realisation of the vector $\boldsymbol{X}$. Note that $\mathrm{g}(\boldsymbol{x}) = 0$ describes the boundary of the limit state, and the inequality $\mathrm{g}(\boldsymbol{x}) > 0$ the safe state of a structure.

If the joint probability density $\mathrm{f}_{\boldsymbol{X}}(\boldsymbol{x}|H_i)$ of basic variables $\boldsymbol{X}$ given situation H_i is known, the conditional probability of failure $\mathrm{P}\{F|H_i\}$ can be then determined [6] using the integral

$$\mathrm{P}\{F|H_i\} = \int_{\mathrm{g}(\boldsymbol{x})<0} \mathrm{f}_X(\boldsymbol{x} \mid H_i)\mathrm{d}\boldsymbol{x} \tag{2}$$

It should be mentioned that the probability $\mathrm{P}\{F|H_i\}$ calculated using equation (2) suffers generally from two essential deficiencies:

- uncertainty in the definition of the limit state function $\mathrm{g}(\boldsymbol{x})$,
- uncertainty in the theoretical model for the density function $\mathrm{f}_{\boldsymbol{X}}(\boldsymbol{x}|H_i)$ of basic variables $\boldsymbol{X}$ [8].

These deficiencies are most likely the causes of the observed discrepancy between the determined probability p_F and actual frequency of failures; this problem is particularly disturbing in case of fire. Yet, the probability requirement $p_F < p_\mathrm{t}$ is generally accepted as a basic criterion for design of structures.

In a risk analysis we need to know not only probability of the structural failure F but probabilities of all events having unfavourable consequences. In general, the situations H_i may cause a number of events E_{ij} (e.g. excessive deformations, full development of the fire). The required conditional probabilities $\mathrm{P}\{E_{ij}|H_i\}$ must be estimated by a separate analysis using various methods, for example the fault tree method or causal networks.

5 ESTIMATION OF CONSEQUENCES

Consequences are possible outcomes of a desired or undesired event that may be expressed verbally or numerically to define the extent of human fatalities and injuries or environmental damage and economic loss [1]. A systematic procedure to describe and/or calculate consequences is called consequence analysis. Obviously, consequences are generally not one-dimensional. However in specific cases they may be simplified and described by several components only, e.g. by human fatalities, environmental damage and costs. At present various costs are usually included only. It is assumed that adverse consequences of the events E_{ij} can be normally expressed by several components $C_{ij,k}$, where the subscript k denotes the individual components (for example the number of lost lives, number of human injuries and damage expressed in a certain currency).

6 ESTIMATION OF RISK

Risk is a measure of the danger that undesired events represent for humans, environment or economic values. Risk is commonly expressed in the probability and consequences of the undesired events. It is often estimated by the mathematical expectation of the consequences of an undesired event. Then it is the product "probability × consequences". However, a more general interpretation of the risk involves probability and consequences in a non-product form. This presentation is sometimes useful, particularly when a spectrum of consequences, with each magnitude having its own probability of occurrence, is considered [2].

The estimation of risk is the process used to produce an estimate of a measure of risk. As already stated above the risk estimation is based on the hazard identification and generally contains the following steps: scope definition, frequency analysis, consequence analysis, and their integration [2]. If there is one-to-one mapping between the consequences $C_{ij,k}$ and the events E_{ij}, then the risk component R_k related to the considered situations H_i is the sum

$$R_k = \sum_{ij} C_{ij,k} \mathrm{P}\{E_{ij} \mid H_i\} \mathrm{P}\{H_i\} \tag{3}$$

If the dependence of consequences on events is more complicated than just one-to-one mapping, then equation (3) will have to be modified. A practical example of equation (3) can be found in [10], where an attempt to estimate the risk due to persistent and fire design situation is presented.

In some cases it is possible to deal with one-component risk R only. Then the subscript k in equation (2.3) may be omitted. Moreover, probability of undesired events may depend on the vector of basic variables $\boldsymbol{X}$. Then the total risk R may be formally written as

$$R = \int C(\mathbf{x}) \mathrm{f}_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{4}$$

where $R(\boldsymbol{x})$ denotes the degree of risk as a function of basic variables $\boldsymbol{X}$, and $\mathrm{f}_{\boldsymbol{X}}(\boldsymbol{x})$ denotes joint probability density function $\boldsymbol{X}$.

7 LOGIC TREES

A number of different logic (decision) trees (fault tree, event tree, cause/consequence chart) have been developed to analyse the risk of a system [11] to [13]. Applications of logic trees significantly improve the completeness and clarity of the engineering work. The use of this kind of tool is widespread in risk analysis and implies some important advantages. Influences of the environment and of human activities can easily be considered simultaneously. Logic trees can also enable the detection of the most effective countermeasures. Furthermore, they can be easily understood by inexperienced persons and therefore can provide very effective communication means between experts and public authorities.

The fault tree can be defined as a logical diagram for the representation of combinations of influences that can lead to an undesired event. When establishing a fault tree, the undesired event constitutes the starting point. Going out from this event, possible causes are to be identified. The possible causes and consequences are to be linked in a logic way, without introducing any loops. Every event that is not a consequence of the previous event has to be considered as an independent variable.

An example of the fault tree shown in Figure 3 describes the failure of a plane frame (indicated at the bottom of Figure 3).

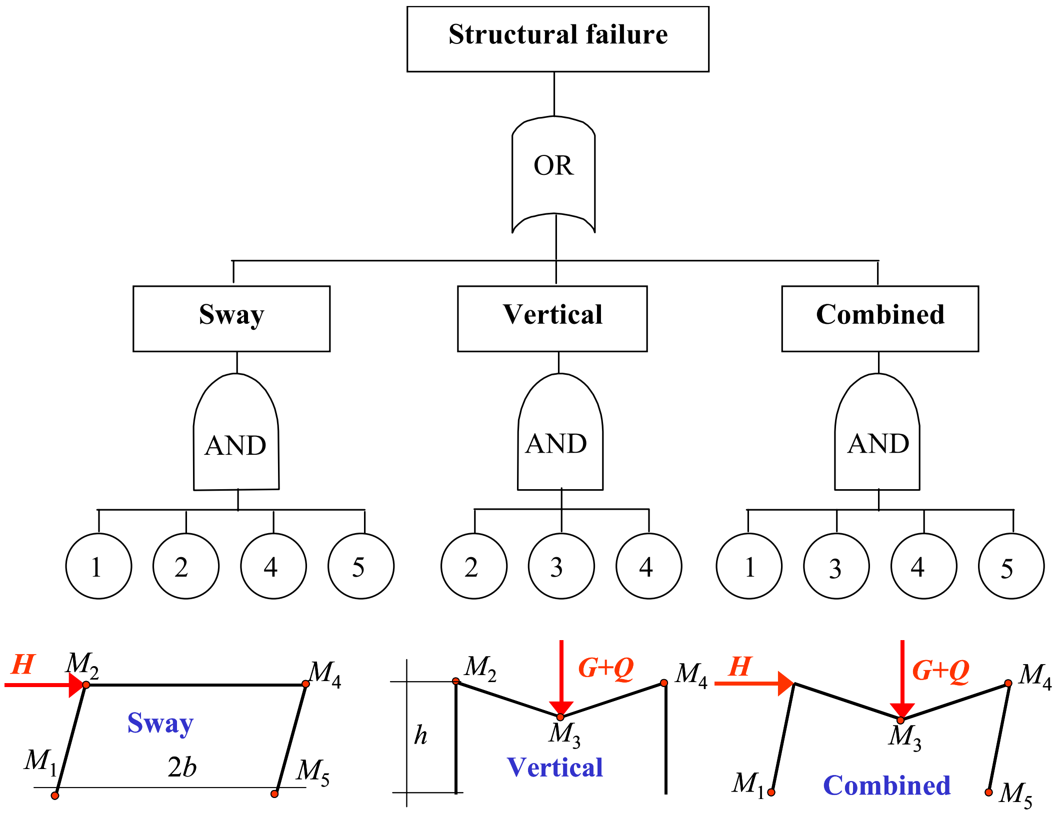

Figure 3. Fault tree describing the failure of a plane frame.

Fault trees can be used to clarify the causes of failures in case that they are unknown. The most common application, however, consists in detecting possible causes of undesirable events before they can occur. Since the fault trees also show the possible consequences of events, they are very useful for the establishment of the most accurate measures for prevention of these events.

An event tree identifies possible subsequent events starting from an initial event. Each path consists of a sequence of events and ends up at the consequence level (for example at structural failure, see Figure 4). The aim of the event tree analysis is to identify possible consequences of an initial event and to calculate probabilities of the occurrence of these consequences corresponding to a different sequence of events.

Simple examples of an event tree describing the collapse of a structure under persistent and fire (accidental) design situation is shown in Figure 4. The probabilities indicated in Figure 4 are illustrative values only (correspond approximately to a 50-year period of an administrative building having the fire compartment area 250 m^2 with a protected steel structure and without sprinklers).

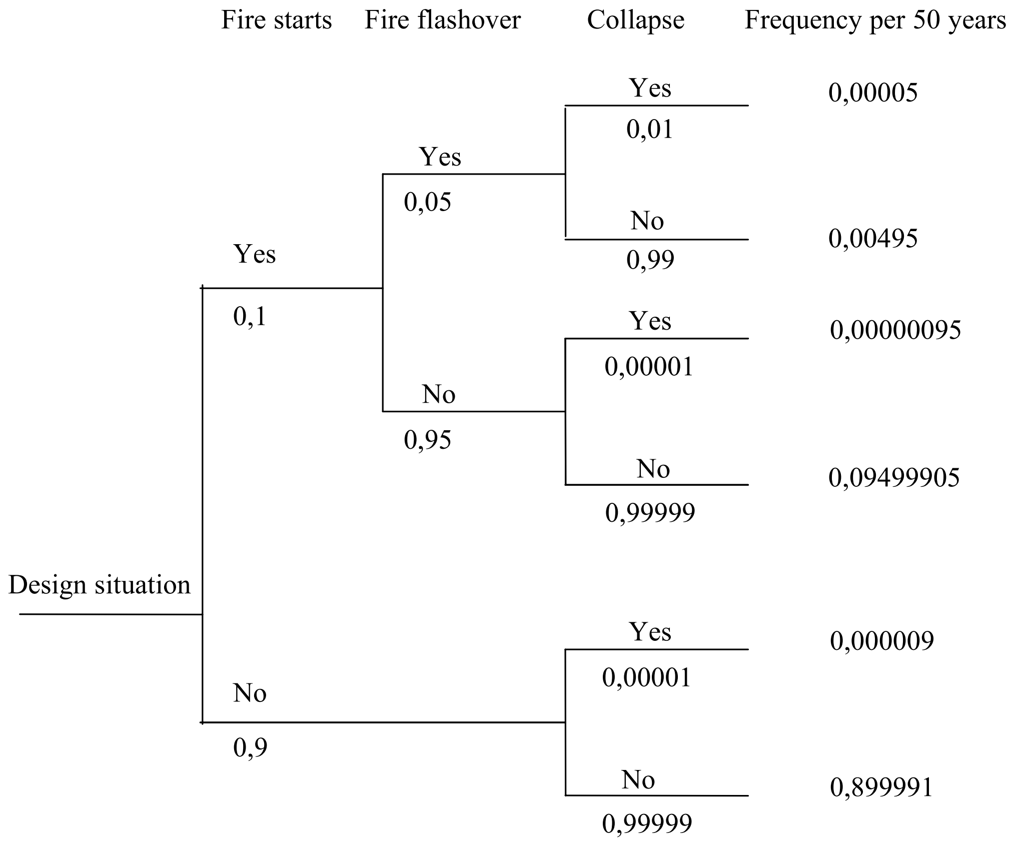

Figure 4. Event tree describing the collapse of a structure under persistent and fire design situation (all data are approximately related to a 50-year period of an administrative building having the fire compartment area 250 m^2 without sprinklers and with a protected steel structure).

Logic trees may be supplemented by the consequences of events; graphical representation of such a tree is called the cause/consequence-chart. The consequence chart corresponds to an event tree with a suitable representation of expected consequences. For example Figure 4 may include consequences linked to each failure probability (frequency per year) of the structural collapse under given conditions. Then the tree may be used for the cause/consequence or risk (utility) analysis.

The simplest form of the cause/consequence consideration is the so-called prior-analysis of the risk (utility) when the basic statistical and probabilistic information is available prior to any decision or activity. The prior analysis is an assessment of the risk associated with different decisions; commonly used for comparing the risks corresponding to different decisions. The posterior decision analysis differs from the prior analysis by considering possible changes in the branching probabilities and/or the consequences due to risk reducing measures, risk mitigating measures and the collection of additional information. The posterior decision analysis may be used to evaluate different additional activities affecting the total risk.

Other important modification of logic trees is known as the pre-posterior decision analysis. The aim of the pre-posterior decision analysis is to identify the optimal decisions with regard to activities that may be performed in the future, e.g. planning of risk reducing activities and/or the collection of new information. An important pre-requisite for the pre-posterior decision analysis is the consideration of future actions that may be applied taking into account the results of the planned activities.

8 BAYESIAN NETWORK

Another promising tool for the risk analysis seem to be Bayesian (believe) causal networks [13,14]. A simple example of the causal network is shown in Figure 5. The network containing only four chance nodes describes the structural failure under persistent and fire design situation similarly as the event tree in Figure 4. Compared with the event tree shown in Figure 4 the network in Figure 5 includes also the effect of sprinklers (node B). Note that the directional arrows in Figure 5 indicate the causal links between interconnected chance nodes.

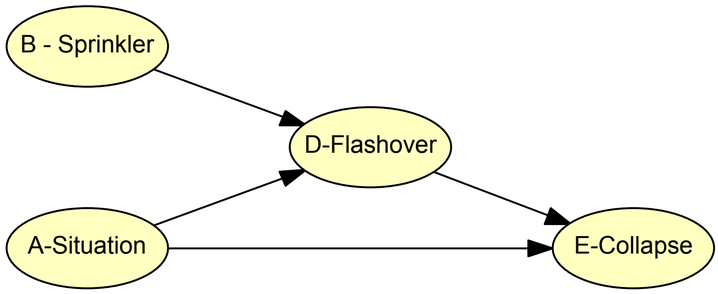

Figure 5. The causal network describing the structural failure under persistent and fire design situation.

The collapse of a structure depends on the probability of persistent and fire situation and on the conditional probabilities of full development of fire, that depends on the ability of sprinklers and on conditional probability of the structural collapse under the conditions given by parents nodes (for example when fire is fully developed - fire flash over). Obviously the causal network representation seems to be much more effective than the event tree version. Moreover each node may have several states. Consequently, the input data are not indicated directly in the graphical representation of the network but are given in the tables of conditional probabilities.

The basic principle of probability calculation used in the Bayesian networks may be illustrated considering the nodes A, B and D of the network in Figure 5. One child node *D* (Fire flashover) is dependent on two parent nodes: *A* (Design situation) and *B* (Sprinklers). If the parents' nodes *A* and *B* have the discrete states A_i and B_j, then the probability of the event D_k (a particular state of the node *D*) is given by the formula

$$\mathrm{P}(D_k) = \sum \mathrm{P}(D_k|\, A_iB_j)\mathrm{P}(A_i)\,\mathrm{P}(B_j) \tag{5}$$

Equation (5) represents the fundamental theoretical tool for analysing the Bayesian network. The input data consist of the probabilities $\mathrm{P}(A_i)$ and $\mathrm{P}(B_j)$, and the conditional probabilities $\mathrm{P}(D_k|A_iB_j)$. These extensive data are based on available statistical evidence, probabilistic analysis or expert assessment (judgement) and are transparently summarised in the tables of conditional probabilities.

Bayesian networks supplemented by decision and utility nodes called influence diagrams [13,14] provide a powerful tool for the risk estimation. In fact the influence diagram is a generalisation of the cause/consequence-chart discussed above. The main features of this tool are illustrated by the example shown in Figure 6, which is an extension of the fundamental task indicated in Figure 5. Figure 6 shows a simplified influence diagram developed recently [15,16] for the risk analysis of buildings under persistent and fire design situation.

The network consists of seven chance nodes numbered 1, 2, 3, 4, 5, 12 and 14, four decision nodes 6, 7, 15 and 16, and six utility nodes 8, 9, 10, 11, 13 and 17. The utility nodes represent the costs of various fire safety measures (nodes 8, 10, 17), damage to the building (nodes 9, 11), and injuries (node 13).

Directional arrows indicating the causal links between the parent and children nodes interconnect the chance, decision and utility nodes. All the causal links must be described by appropriate input data (conditional probabilities or utility units) linked to assumed states of the nodes. For example the utility nodes (except the utility node 13) are directly dependent on the size of the building (node 15). The utility node 13, describing the cost of injury, is affected by the size of the building through the number of endangered persons represented by chance node 14. These data are sometimes difficult to specify, and an expert assessment has often to be often.

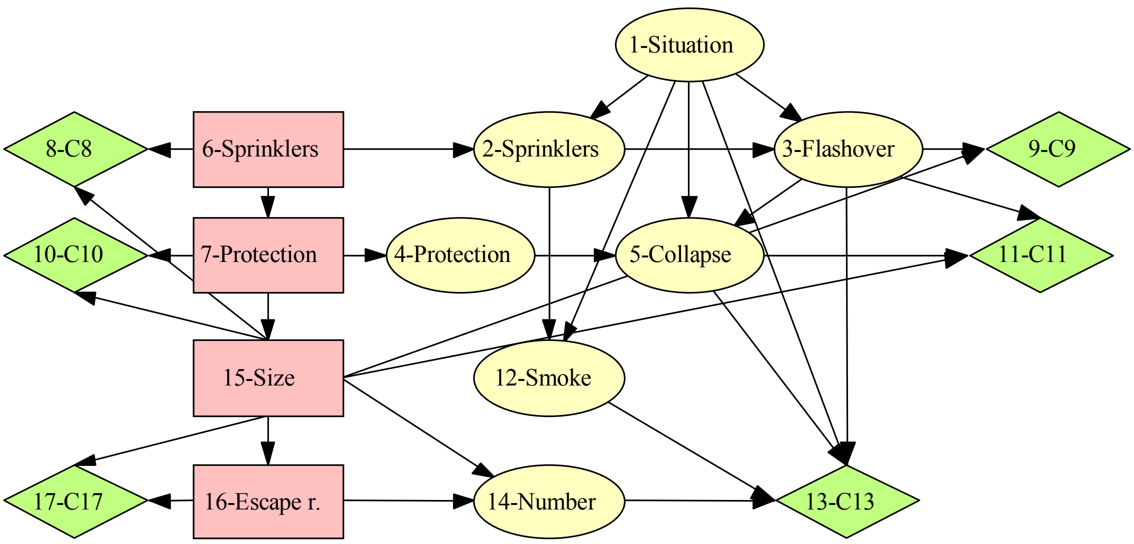

Figure 6. The Bayesian network describing a structure under normal and fire design situations.

The chance nodes 1, 2, 3, 4, 5, 12 and 14 represent alternative random variables having two or more states. The node 1-Situation describes the probability of fire start $p_{\text{fi,s}} = \text{P}(H_2)$ and the complementary probability $1- p_{\text{fi,s}}$ of normal situation H_1. The chance node 2-Sprinklers describes the functioning of sprinklers provided that the decision (node 6) is positive; the probability of the active state of the sprinklers given the fire start is assumed to be very high, for example 0,999. The chance node 3-Flashover has two states: the design situation H_3 (fire design situation without flashover) and H_4 (fire design situation with flashover when the fire is fully developed).

When sprinklers are installed, the flashover in a compartment of 250 m^2 has the positive state with the conditional probability 0,002; if sprinklers are not installed then $\text{P}\{H_4|H_2\}$ = 0,066 [15,16]. It is assumed that with the probabilities equal to the squares of the above values the fire will flash over the whole building, thus the values 0,000004 and 0,0044 are considered for the chance node 3. The chance node 4-Protection (introduced for formal computational reasons) has identical states as the decision node 7-Protection. The chance node 5-Collapse represents the structural failure that is described by the probability distribution linked to three children nodes (1,3,4). This situation can hardly be modelled using a decision tree. Note that the probability of collapse in the case of fire but not flashover may be smaller than in a persistent situation, due to the lower imposed load.

9 DECISION-MAKING

The decision-making is generally based on the process of the risk acceptance and option analysis (see Figure 1) that is sometimes referred to as the risk evaluation. The risk acceptance is based on various criteria of risk that are reference points against which the results of the risk analysis are to be assessed. The criteria are generally based on regulations, standards, experience, and/or theoretical knowledge used as a basis for the decision about the acceptable risk. Acceptance criteria and the criteria of risk may be sometimes distinguished [1]. Various aspects may be considered, including cultural, social, psychological, economical and other aspect [6], [17], [18] and [19]. Generally the acceptance criteria may be expressed verbally or numerically [6].

Assuming for example that the acceptance limits $C_{k,\mathrm{d}}$ for the components C_k are specified, then it is possible to design the structure on the basis of acceptable risks using the criterion $C_k < C_{k,\mathrm{d}}$, which may supplement the probability requirement $p_F < p_{\mathrm{t}}$.

It should be noted that various levels of risk might be recognized, for example acceptable risk, tolerable risk, and objective risk [6] (see the definitions of theses terms in [2]). It is a remarkable fact that the public seems to be generally better prepared to accept certain risks than to stand for specified probabilities of failure [17].

10 THE IMPLIED COST OF AVERTING A FATALITY

The consequences may generally include economic as well as social and environmental costs [17,18,19]. An example is provided by the influence diagram shown in Figure 6 used to assess the risk of a building due to fire. Thus, in order to compare all possible consequences it is necessary to express all consequences in terms of a single unit. This seems to be an extremely difficult task. One of the possible approaches is represented by the concept of the Implied Cost of Averting a Fatality ICAF or Life Quality index LQI [19]. Table 1 shows values of the cost ICAF for selected countries adopted from [19]. It appears that the cost ICAF may be estimated to about 1 to 3 million of USD.

Table 1. The Implied Cost of Averting a Fatality – ICAF(Δe), financial data in PPP US$ (1999) obtained from UN-HDR 2001, World Bank.

Country	g- annual income	e- life time	2 w- working part of e	ICAF(Δe) [$\times 10^6$]
US	34000	77	0.15	2.6
Japan	26000	81	0.15	2.1
Germany	25000	77	0.125	1.9
UK	22000	77	0.125	1.7
Czech Republic	8000	75	0.15	0.6
Mexico	8800	72	0.15	0.6
South Africa	9100	55	0.15	0.5
Colombia	5900	70	0.15	0.4
China	3900	70	0.15	0.3
India	2400	63	0.15	0.1
Nigeria	800	47	0.18	0.04

The Implied Cost of Averting a Fatality *ICAF* can be expressed as

$$ICAF(\Delta e) = g\left(1-(1+\frac{\Delta e}{e})^{1-\frac{1}{w}}\right)\Delta e \tag{6}$$

where symbols *g, e* and *w* are defined in Table 1. However, the concept of the Implied Cost of Averting a Fatality described by equation (6) is just one of possible approaches to the complex problem of evaluating social consequences. At present further intensive investigation is expected.

11 CONCLUDING REMARKS

Risk is commonly estimated by the mathematical expectation of the consequences of an undesired event that often leads to the product "probability × consequences". As a rule the risk of civil engineering systems is a multidimensional quantity having several components.

The risk analysis is based on the hazard identification and generally contains the following steps: the scope definition, hazard identification, definition and modelling of hazard scenarios, estimation of probabilities, estimation of consequences, estimation of risk and decision-making.

The most important contribution of the risk analysis and assessment consists in the systematic consideration of various consequences. Several techniques are available at present: the decision trees, the Bayesian belief networks and influence diagrams. Available experience indicates that the Bayesian belief networks provide a transparent, logical and effective tool for analysing engineering systems. It should however be underlined that any analysis of an engineering system is always dependent on the assumed input data, often of a very uncertain nature. The input data should be estimated with due regard to the specific technological and economic conditions of a given system. In particular, the economic, social and environmental consequences of adverse events should be further investigated.

It appears that the methods of risk analysis and assessment may significantly contribute to further improvement of current engineering design. The remarkable fact that the public is better prepared to accept certain risks than to stand for specified probabilities of failure will make the application of the risk assessment easier. It is therefore anticipated that in the near future probabilistic methods in engineering will be supplemented by criteria for acceptable risks.

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NOTATION

A_i States of node A
B_j States of node B
D_k States of node D
C_{ij} Consequences of events E_{ij} (utility, cost, damage, injuries)
R The total expected risk
R_k The risk component
E_{ij} Events
H_i Hazard situation i.
$\mathrm{P}(F/H_i)$ Probability of failure F given situation H_i
e Expected life-time
g Annual income
w Working part of e
$\mathrm{g}(\boldsymbol{x})$ Performance (limit state) function.
p_F Probability of failure F.
p_d Target probability of failure.
p_f Probability $\mathrm{P}(F|H_2)$ of structural failure during fire.
$p_\mathrm{fi,s}$ Probability of fire start $\mathrm{P}(H_2)$.
$\boldsymbol{x}$ Generic point of the vector of basic variables.
$\boldsymbol{X}$ Vector of basic variables.
β Reliability index.
$\varphi_{\boldsymbol{X}}(\boldsymbol{x})$ Probability density function of the vector of basic variables $\boldsymbol{X}$.
$\Phi_\mathrm{N}^{-1}(p_F)$ Inverse distribution function of a standardized normal variable.

ANNEX A - BASIC STATISTICAL CONCEPTS AND TECHNIQUES

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Summary

Elementary concepts and techniques of the theory of probability and mathematical statistics required for understanding of basic reliability methods are reviewed and illustrated by a number of numerical examples. Computational procedures for determination of sample characteristics, fractiles of common theoretical models and estimates for fractiles based on small samples can be applied using the attached MATHCAD sheets.

1 INTRODUCTION

1.1 Background materials

Elementary concepts and techniques of the theory of probability and mathematical statistics applicable to civil engineering are available in a number of standards [1 to 5], background materials [6, 7, 8], software products [9, 10, 11] and books [12 to 24]. Additional information may be found in the extensive literature listed in the books [12, 13] and others. In particular, documents developed by JCSS [6, 7] and recently published handbook [8] are closely related to the statistical techniques described in this text.

1.2 General principles

The theory of structural reliability is based on a general principle that all the basic variables are considered as random variables having appropriate type of probability distribution. Different types of distributions should be used for description of actions, material properties and geometric data. Prior theoretical models of basic variables and procedures for probabilistic analysis are indicated in JCSS documents. Sample characteristics are used as estimates of population parameters. In addition the population fractiles must be often assessed using small samples. MATHCAD sheets that supplement described computational procedures can be effectively used in practical applications.

2 POPULATION AND SAMPLES

2.1 General

Actions, mechanical properties and geometric data are generally described by random variables (mainly by continuous variables). A random variable *X*, (e.g. concrete strength), is such a variable, which may take each of the values of a specified set of values (e.g. any value from a given interval), with a known or estimated probability. As a rule, only a limited number of observations, constituting a random sample x_1, x_2, x_3,..., x_n of size *n* taken from a population, is available for a variable *X*. Population is a general statistical term used for the totality of units under consideration, e.g. for all concrete produced under specified conditions within a certain period of time. The aim of statistical methods is to make decisions concerning the properties of the population using the information derived from one or more random samples.

2.2 Sample characteristics

A sample characteristic is a quantity used to describe the basic properties of a sample. The three basic sample characteristics, which are most commonly used in practical applications, are:

- the mean m representing the basic measure of central tendency;
- the variance s^2 describing the basic measure of dispersion; and
- the coefficient of skewness ω giving the basic measure of asymmetry.

The sample mean m (an estimate of the population mean) is defined as the sum

$$m = (\Sigma\, x_i) / n \tag{2.1}$$

with the summation being extended over all the n values of x_i.

The sample variance s^2 (an estimate of the population variance), is defined as:

$$s^2 = (\Sigma\, (x_i - m)^2) / (n - 1) \tag{2.2}$$

the summation being again extended over all values x_i. Sample standard deviation s is the positive square root of the variance s^2.

The sample coefficient of skewness ω (an estimate of the population skewness) characterising asymmetry of the distribution is defined as

$$\omega = [n\, (\Sigma\, (x_i - m)^3) / (n-1) / (n-2)] / s^3 \tag{2.3}$$

Thus, the coefficient of skewness is derived from the central moment of order 3 divided by s^3. If the sample has more distant values to the right from the mean than to the left, the distribution is said to be skewed to the right or to have a positive skewness. If the reverse is true, it is said to be skewed to the left or to have a negative skewness.

In some cases two different samples may be taken from one population and their combination is needed. If the original data are not available, then the characteristics of combined sample may be determined using the characteristics of both samples. If the sample sizes are n_1, n_2, the means m_1, m_2, standard deviations s_1, s_2 and skewnesses ω_1, ω_2, then the combined sample of the size $n = n_1+n_2$ has the characteristics

$$m = \frac{n_1 m_1 + n_2 m_2}{n}$$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n} + \frac{n_1 n_2}{n^2}(m_1 - m_2)^2 \tag{2.4}$$

$$\omega = \frac{1}{s^3}\left[\frac{n_1 s_1^3 \omega_1 + n_2 s_2^3 \omega_2}{n} + \frac{3 n_1 n_2 (m_1 - m_2)(s_1^3 - s_2^3)}{n^2} - \frac{n_1 n_2 (n_1 - n_2)(m_1 - m_2)^3}{n^3}\right]$$

Another important characteristic describing the relative dispersion of a sample is the coefficient of variation v, defined as the ratio of standard deviation s to the mean m

$$v = s / m \tag{2.5}$$

The coefficient of variation v can be effectively used only if the mean m differs from zero. When the mean is much less than the standard deviation, then the standard deviation rather then the coefficient of variation should be considered as a measure of the dispersion. The coefficient of variation v is often used as a measure of production quality; for concrete strength may be expected within a broad range from 0,05 up to 0,20, for structural steel from 0,07 to 0,10.

2.3 Distribution function

Probability distribution is a term generally used for any function giving the probability that a variable X belongs to a given set of values. The basic theoretical models used to describe the probability distribution of a random variable may be obtained from a random sample by increasing the sample size or by smoothing either the frequency distribution or the cumulative frequency polygon.

An idealisation of a cumulative frequency polygon is the distribution function $\Phi(x)$ giving, for each value x, the probability that the variable X is less than or equal to x:

$$\Phi(x) = \mathrm{P}\ (X \leq x) \tag{2.6}$$

A probability density function $\varphi(x)$ is an idealisation of a relative frequency distribution. It is formally defined as the derivative (when it exists) of the distribution function:

$$\varphi(x) = \mathrm{d}\Phi(x)\ /\ \mathrm{d}x \tag{2.7}$$

Note that Appendix 1 to this Chapter provides a review of selected theoretical models of continuous random variables that are most frequently used in reliability analysis of civil structures.

Example 2.1.

A continuous random variable, which may attain equally likely any point x within a two-sided interval $\langle a, b \rangle$ (each point x has the same probability density $\varphi(x)$) is described by a so-called uniform distribution shown in Figure 2.1.

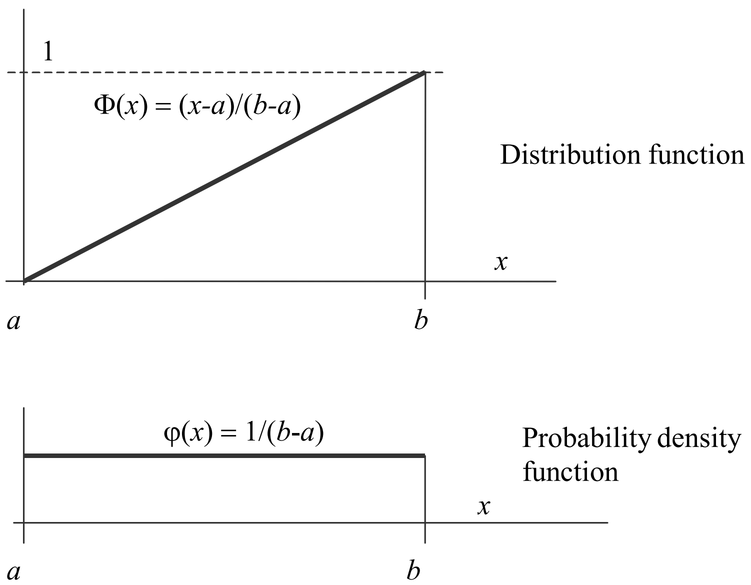


Figure 2.1. Uniform distribution

The uniform distribution is a basic type of distribution used not only in simulation procedures but also in theoretical modelling of some actions and geometric data. Shapes of the distribution function $\Phi(x)$ and probability density function $\varphi(x)$ for the uniform distribution are shown in Figure 2.1. We can easily observe that it is a general property of the probability density function that the probability of a set of all values of any random variable is equal to 1

$$\int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x = \int_{a}^{b} \varphi(x)\mathrm{d}x = 1 \tag{2.8}$$

Thus, the surface bounded by the horizontal axis x and the curve of the density function $\varphi(x)$ has the area equal to unity.

2.4 Population parameters

The population parameters are quantities used in describing the distribution of a random variable, as estimated from one or more samples. As in the case of random samples, three basic population parameters are commonly used in practical applications:

- the mean μ representing the basic measure of central tendency;
- the variance σ^2 as the basic measure of dispersion; and
- the coefficient of skewness ω giving the degree of asymmetry.

The population mean μ, for a continuous variable X having the probability density $\varphi(x)$, is defined as

$$\mu = \int x\, \varphi(x)\mathrm{d}x \tag{2.9}$$

the integral being extended over the interval of variation of the variable X. The population variance σ^2, for a continuous variable X having the probability density function $\varphi(x)$, is the mean of the squared deviation of the variable from its mean:

$$\sigma^2 = \int (x - \mu)^2\, \varphi(x)\mathrm{d}x \tag{2.10}$$

The population standard deviation σ is the positive square root of the population variance σ^2.

The population coefficient of skewness, characterising asymmetry of the distribution, is defined as

$$\omega = \int (x - \mu)^3\, \varphi(x)\mathrm{d}x \,/\, \sigma^3 \tag{2.11}$$

Another population parameter based on the fourth order moment is called kurtosis ε.

$$\varepsilon = \int (x - \mu)^4\, \varphi(x)\mathrm{d}x \,/\, \sigma^4 - 3 \tag{2.12}$$

Note that for normal distribution (described in Section 3.1) the kurtosis ε defined by equation (2.12) is zero. However, this parameter is used mainly in theoretical consideration.

Another important parameter of the population is the coefficient of variation V defined similarly as the sample coefficient of variation

$$V = \sigma / \mu \tag{2.13}$$

The same restriction on the practical use of V applies as in the case of samples.

Geometrically μ is actually the x coordinate of the centre of gravity of the area bounded by the horizontal axis x and the curve of density function $\varphi(x)$. Figure 2.2 shows an example of probability density function of lognormal distribution illustrating the geometric interpretation of the mean μ and standard deviation σ.

The measure of dispersion of a random variable X relative to the mean μ is given by the central moment of the second order (moment of inertia) of the area, and standard deviation

σ is therefore the centroidal radius of gyration around the mean μ of the area bounded by the horizontal axis x and the curve of probability density function $\varphi(x)$.

A very important population characteristic is the fractile x_p. If X is a continuous variable and p is a probability (a real number between 0 and 1), the p-fractile x_p is the value of the variable X for which the probability that the variable X is less than or equal to x_p is p, and hence, for which the distribution function $\Phi(x_p)$ is equal to p. Thus,

$$P\ (X \leq x_p) = \Phi(x_p) = p \tag{2.14}$$

In civil engineering the probabilities p = 0,001; 0,01; 0,05 and 0,10 are used most frequently. The probability p is often written as a percentage (e.g. p = 0,1 %; 1 %; 5 %; 10 %). If this is done, then x_p is called a percentile, for example the 5th percentile is used when p = 5 %. If p=50 %, then x_p is called the median. More details about the fractiles of continuous variables are given in the following sections.

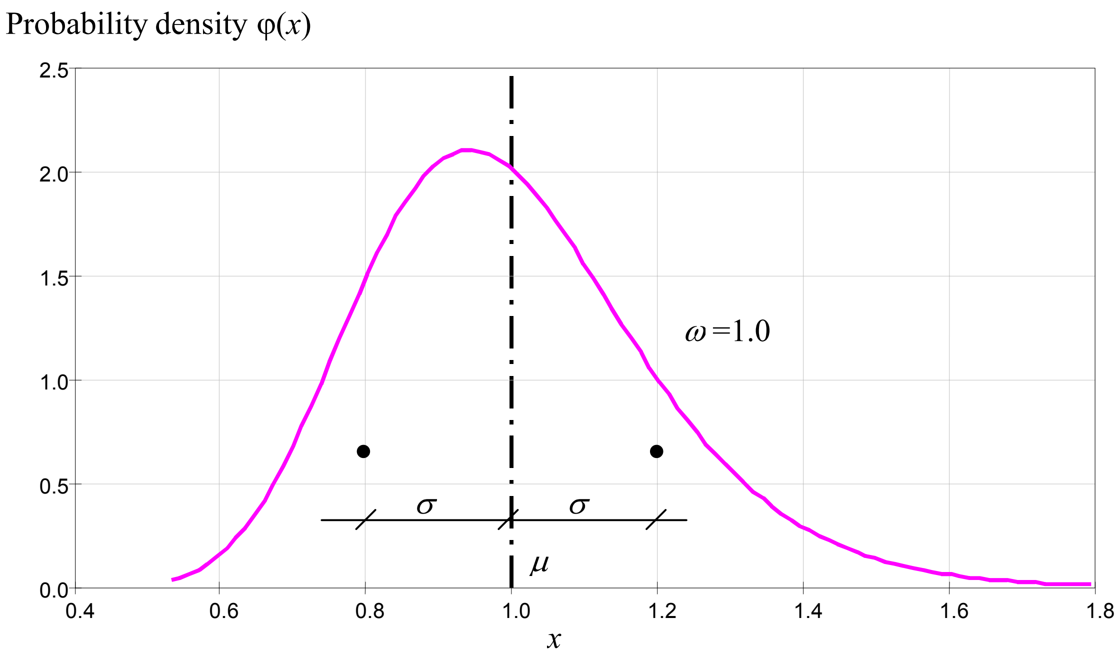


Figure 2.2. Geometric illustration of the mean μ and standard deviation σ

Example 2.2.

Parameters of the uniform distribution from example 2.1 may be derived using equations (2.9) to (2.13) as

$$\mu = (a+b)/2,\ \sigma = (b\text{-}a)/\sqrt{12}\ ,\ \omega = 0,\ \varepsilon = -\ 2{,}96,\ V = (b\text{-}a)/((a+b)\sqrt{3}\ )$$

The skewness of a uniform distribution is zero, kurtosis is negative (independent of the bounds a and b). Obviously the distribution is symmetric as the values of the random variable are distributed uniformly. If the lower bound of the distribution is zero, a = 0 (which is sometimes assumed in practical applications), then

$$\mu = 0{,}5\ b,\ \sigma = 0{,}289\ b,\ \omega = 0,\ \varepsilon = -\ 2{,}96,\ V = 0{,}577$$

Let us note that the coefficient of variation V in this case (when a = 0) is independent of b and its value is relatively high (V = 0,577).

3 SELECTED MODELS OF RANDOM VARIABLES

3.1 Normal distribution

Most frequently used models of continuous random variables that are applied in reliability analysis of civil structures are reviewed in Appendix 1 of this Chapter. From a practical and theoretical point of view the most important type of distribution of a continuous random variable is the normal (Laplace-Gauss) distribution. Symmetric normal distribution of a variable X is defined on an unlimited interval $-\infty < x < \infty$ (which can be undesirable in some practical applications) and depends on two parameters only – on the mean μ and on the standard deviation σ. Symbolically it is often denoted as $N(\mu,\sigma)$.

The normal distribution is frequently used as a theoretical model of various types of random variables describing some loads (self-weight), mechanical properties (strengths) and geometrical properties (outer dimensions). It is convenient for symmetric random variable with a relatively low variance (coefficient of variation V < 0,3). It fails when used for asymmetric variables with great variance and skewness ω > 0,5.

The probability density function of a normal random variable X with a mean μ and standard deviation σ is given by the exponential expression

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right] \tag{3.1}$$

Skewness ω and kurtosis ε are zero for a normal distribution.

Tables for normal distributions are commonly available [12, 13] for probability density function $\varphi(u)$ and distribution function $\Phi(u)$ of a standardized variable U, which is defined by a general transformation relation (used for any type of distribution)

$$U = \frac{X-\mu}{\sigma} \tag{3.2}$$

The standardized random variable U has a zero mean and variance (standard deviation) equal to one; symbolically it is often denoted as N(0, 1).

The probability density function of the standardized random variable U is then given as a function of u

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2} \right) \tag{3.3}$$

The probability density function of a normal and lognormal distribution with a coefficient of skewness ω = 1,0 (described in the next section 3.2) of the standardized random variable u is shown in Figure 3.1.

Note that the probability density function of the standardized normal distribution is plotted in Figure 3.1 for u in the interval <–3,+3>, which covers the standardised variable U with a high probability of 0,9973 (in engineering practice this interval is often called interval $\pm 3\sigma$).

3.2 Lognormal distribution

Generally one-sided limited asymmetric lognormal distribution is defined on a limited interval $x_0 < x < \infty$ or $-\infty < x < x_0$. Therefore it eliminates one of the undesirable properties of the normal distribution. A lognormal distribution is generally dependent on three parameters. Commonly the moment parameters are used: mean μ_X, standard deviation σ_X and skewness ω_x. If the skewness ω_x is unknown or uncertain, the lower or upper bound x_0 is used.

Random variable X has a lognormal (general three-parametric) distribution if the transformed random variable

$$Y = \ln |X - x_0| \tag{3.4}$$

has a normal distribution. In this relation x_0 denotes the lower or upper limit of distribution of a variable X, which depends on skewness ω_x. If the variable has a mean μ_x and standard deviation σ_x, then the lower or upper limit can be expressed as

$$x_0 = \mu_X - \sigma_X / c \tag{3.5}$$

where the coefficient c is given by the value of skewness ω_X according to the relation

$$\omega_X = c^3 + 3c^3 \tag{3.6}$$

from which follows an explicit relation for c

$$c = \left[\left(\sqrt{\omega_X^2 + 4} + \omega_x\right)^{1/3} - \left(\sqrt{\omega_X^2 + 4} - \omega_X\right)^{1/3}\right] 2^{-1/3} \tag{3.7}$$

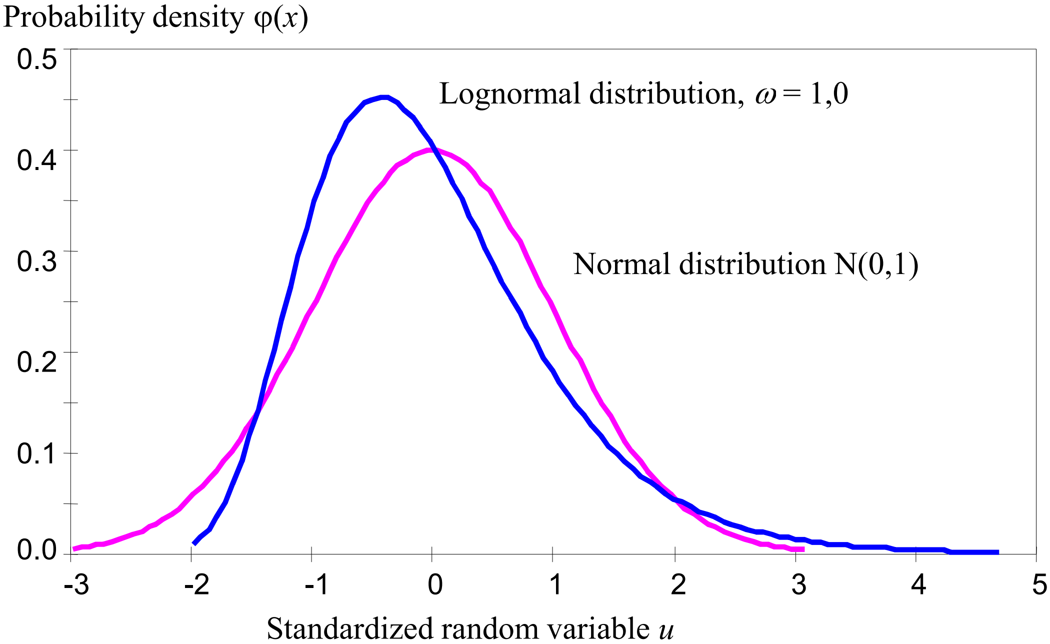


Figure 3.1. Normal and lognormal distribution (skewness $\omega = 1{,}0$)

The dependence of the limit x_0 on the coefficient c is obvious from Table 3.1 in which the lower bound $u_0 = -1/c$ of the standardised random variable $U=(X-\mu_X)/\sigma_X$ are given for selected values of the coefficient of skewness $\omega_X \geq 0$. For $\omega_X \leq 0$ values of u_0 with an inverse sign (i.e. positive) are considered. A lognormal distribution with the skewness $\omega_x = 0$ becomes a normal distribution ($u_0 = -1/c \rightarrow \pm\infty$).

Table 3.1. The lower limit $u_0 = -1/c$ for selected values of coefficient of skewness $\omega_X \geq 0$.

ω_X	0	0,5	1,0	1,5	2,0
$u_0 = -1/c$	-∞	-6,05	-3,10	-2,14	-1,68

When creating a theoretical model it is therefore possible to consider, besides the mean μ_X and standard deviation σ_X, the skewness ω_X or alternatively the lower or upper bound of distribution x_0. Generally the former possibility is preferred because more credible information is available about the coefficient of skewness, which better characterises the overall distribution of the population (particularly of large populations) compared to the lower or upper bounds.

The probability density function and distribution function of the general three parameter lognormal distribution may be obtained from well known normal distribution using modified (transformed) standardised variable u' obtained from the original standardised random variable $u = (x-\mu_X)/\sigma_X$ as

$$u' = \frac{\ln\left(\left|u+\frac{1}{c}\right|\right)+\ln\left(c\sqrt{1+c^2}\right)}{\sqrt{\ln(1+c^2)}} \tag{3.8}$$

where (as above) $u = (x-\mu_X)/\sigma_X$ denotes the original standardised variable. The probability density function $\varphi_{LN,U}(u')$ and the distribution function $\Phi_{LN,U}(u') = \Phi_{LN,X}(x)$ of the lognormal distribution are then given as

$$\varphi_{LN,U}(u') = \frac{\varphi(u)}{\left(\left|u+\frac{1}{c}\right|\right)\sqrt{\ln(1+c^2)}} \tag{3.9}$$

$$\Phi_{LN,X}(x) = \Phi_{LN,U}(u') = \Phi(u) \tag{3.10}$$

where $\varphi(u)$ and $\Phi(u)$ denote the probability density and distribution function of the standardised normal variable.

A special case is the popular lognormal distribution with a lower bound at zero (x_0 = 0), which like the normal distribution, depends on two parameters only – the mean μ_X and the standard deviation σ_X (symbolically it is denoted LN(μ, σ)). In such a case it follows from equations (3.5) that the coefficient c is equal to the coefficient of variation V_X. It further follows from equation (3.6) that the skewness ω_X of the lognormal distribution with a lower bound at zero is given by the value of the coefficient of variation V_X as

$$\omega_X = 3V_X + V_X^3 \tag{3.11}$$

Thus the lognormal distribution with the lower bound at zero (x_0 = 0) always has a positive skewness, which may have relatively high value (greater than 0,5); e.g. for the coefficient of variation equal to 0,30 a coefficient of skewness V_x = 0,927 obtained from relation (3.11). Applications of the lognormal distribution with the lower limit at zero (x_o = 0) can thus lead to unrealistic theoretical models (usually underestimating the occurrence of negative and overestimating the occurrence of positive deviations from the mean), particularly for higher values of coefficient of variation V_X. Although the occurrence of negative values can also be undesirable (unrealistic for most mechanical quantities), it is usually negligible from a practical point of view.

Example 3.1.

Reinforcement cover layer of a reinforced concrete cross-section X has a mean μ = 25 mm and standard deviation σ = 10 mm. The probability density function $\varphi(x)$ for a normal distribution and for a lognormal distribution with a lower limit at zero is shown in Figure 3.2.

It follows from Figure 3.2 that the normal distribution leads to occurrence of negative values of the reinforcement cover layer, which obviously does not correspond to reality. On the other hand, the lognormal distribution with lower limit at zero overestimates the occurrence of positive deviations of the cover layer, which may not be realistic either and can further lead to unfavourable influences on the strength of the cross-section. The overestimation of occurrence of extreme positive deviations corresponds to a high skewness ω = 1,36 of the lognormal distribution, which follows from equation (3.11). The available experimental data on the concrete cover indicate that the skewness of the distribution is around $\omega \approx 0{,}5$, in most cases $\omega < 1{,}0$.

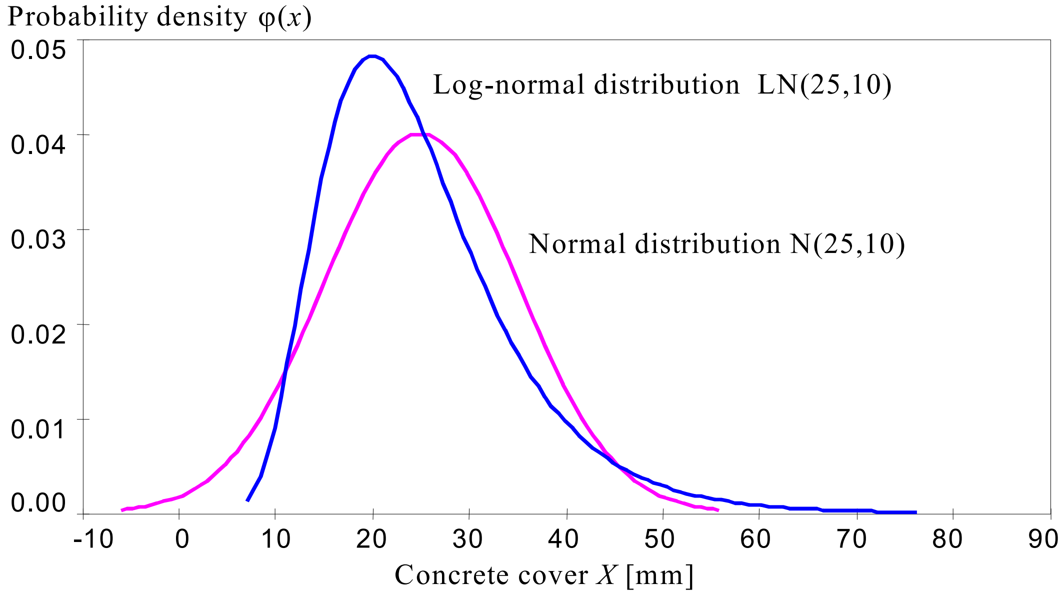


Figure 3.2. Probability density function for the concrete cover

The lognormal distribution is widely applied in the theory of reliability. It is used as a model for various types of random variables describing some loads (self-weight of some materials), mechanical properties (strengths) as well as geometrical data (inner and outer dimensions of cross-sections). It can be used for general asymmetric random variables with both positive and negative skewness. The lognormal distribution with lower limit at zero (x_0 = 0) is very often used for description of mechanical properties (strengths) of various materials (concrete, steel, masonry).

3.3 Gamma distribution

Another popular type of one-side limited distribution is the type III Pearson distribution. Its detailed description is e.g. in the book [13]. A special case of the type III Pearson distribution with lower limit at zero is the gamma distribution. The probability density function of this important distribution is dependent on two parameters only: on the mean μ and standard deviation σ. To simplify the notation two auxiliary parameters λ and k are often used

$$\varphi(x) = \frac{\lambda^k x^{k-1} \exp(-\lambda x)}{\Gamma(k)}, \lambda = \frac{\mu}{\sigma^2}, k = \left(\frac{\mu}{\sigma}\right)^2 \tag{3.12}$$

$\Gamma(k)$ is the gamma function of parameter *k*. For the moment parameters of the gamma distribution it holds that

$$\mu = \frac{k}{\lambda}, \sigma = \frac{\sqrt{k}}{\lambda}, \omega = \frac{2}{\sqrt{k}} = \frac{2\sigma}{\mu} = 2V \tag{3.13}$$

The curve is bell shaped for $k > 1$, i.e. for skewness $\omega < 2$ (in the inverse case it is a decreasing function of x). For $k \to \infty$, the gamma distribution approaches the normal distribution with parameters μ and σ.

The gamma distribution is applied similarly as the lognormal distribution with lower bound at zero. However, it varies from the lognormal distribution by its skewness, which is equal to twice the coefficient of variation ($\omega = 2V$) and is thus lower than the skewness of lognormal distribution, which is more than 50% higher (according to equation (3.11) it is $\omega_X = 3V_X + V_X^3$). That is the reason why the gamma distribution is more convenient for describing some geometrical quantities and variable action that do not have a great skewness.

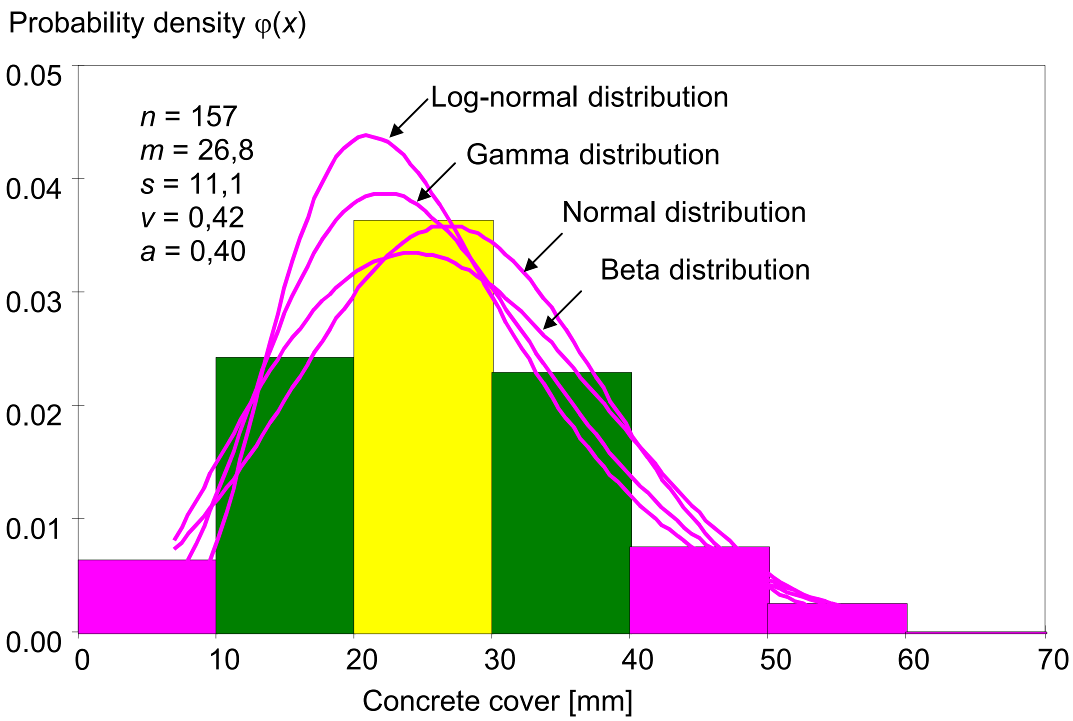


Figure 3.3. Histogram and theoretical models for concrete cover of reinforcement

Example 3.2.

A sample of the size $n = 157$ experimental results of concrete cover of reinforcement measurements has these characteristics: $m = 26{,}8$ mm, $s = 11{,}1$ mm and $v = 0{,}42$. It is a relatively large sample, which can be used for the assessed skewness (furthermore a long-term experience is available). A histogram of the obtained values and theoretical models of normal distribution, lognormal distribution with origin at zero, gamma distribution and beta distribution are shown in Figure 3.3, with help of which the appropriateness of the individual models can be considered.

According to Figure 3.3 it seems that the gamma distribution describes the histogram of obtained results better than the normal and lognormal distribution. But also the both-side limited beta distribution (described in the following Section 3.4) seems to be an appropriate model. However, to choose an appropriate theoretical model for describing variables of interest is a complicated task, which can be treated in theoretical way. Information about some methods of mathematical statistics (about the so-called goodness of fit tests) can be found in the textbook [4] and in specialised literature [12, 13]. In this book some practical aspects and procedures will be indicated only.

3.4 Beta distribution

An interesting type of distribution is the so-called beta distribution (also called Pearson's type I curve), which is defined on a both-side limited interval $<a, b>$ (but this interval can be arbitrarily extended and the distribution then approaches the normal distribution). Generally it is dependent on four parameters and it is used mainly in those cases when it is evident that the domain of the random variable is limited on both sides (some actions and geometrical data, e.g. weight of a subway car, fire load intensity, concrete cover of reinforcement in a reinforced concrete cross-section). The principal difficulty in practical application is the need to estimate all the four parameters, for which credible data may not be available.

The beta distribution is usually written in the form

$$\varphi(x) = \frac{(x-a)^{c-1}(x-b)^{d-1}}{B(c,d)(b-a)^{c+d-1}} \tag{3.14}$$

For the lower and upper limit of distribution it holds

$$a = \mu - c\, g\, \sigma,\ b = \mu + d\, g\, \sigma,\ g = \sqrt{\frac{c+d+1}{cd}} \tag{3.15}$$

where g is an auxiliary parameter. From equations (3.15), relations for parameters c and d can be derived

$$c = \frac{\mu - a}{b-a}\left(\frac{(\mu-a)(b-\mu)}{\sigma^2} - 1\right),\ d = \frac{b-\mu}{b-a}\left(\frac{(\mu-a)(b-\mu)}{\sigma^2} - 1\right) \tag{3.16}$$

For the moment parameters of the beta distribution it holds that

$$\mu = \frac{a+(b-a)c}{(c+d)},\ \sigma = \frac{(b-a)}{(cg+dg)} \tag{3.17}$$

$$\omega = \frac{2g(d-c)}{(c+d+2)},\ \varepsilon = \frac{3g^2(2(c+d)^2 + cd(c+d-6))}{(c+d+2)(c+d+3)} - 3 \tag{3.18}$$

Note that skewness ω and kurtosis ε are dependent only on the parameters c and d (they are independent of the limits a and b). That is why the parameters c and d are called shape parameters. In practical applications the distribution is used for $c > 1$ and $d > 1$ (otherwise the curve is J or U shaped), for $c = d = 1$ it becomes a uniform distribution, for $c = d = 2$ it is the so-called parabolic distribution on the interval $<a, b>$. When $c = d$, the curve is symmetric around the mean. When $d \to \infty$, the curve becomes the type III Pearson distribution (see Section 3.3). If $c = d \to \infty$, it approaches the normal distribution. Depending on the shape parameters c and d the beta distribution thus covers various special types of distributions. The location of the distribution is given by parameters a and b.

The beta distribution can be defined in various ways. If the parameters a, b, c and d are given, then the moment parameters μ, σ, ω and ε can be assess using equations (3.17) to (3.18). In practical applications however, two other combinations of input parameters are often applied:

1. The input parameters are μ, σ, a and b. The remaining parameters c and d can be assessed from equations (3.16), the parameters ω and ε from equations (3.18).
2. The input parameters are μ, σ, ω and one of the limits a (for $\omega > 0$) or b (for $\omega < 0$); the parameters b (or a), c and d can be assessed using equations (3.16) to (3.17).

In practical applications the distribution with lower limit $a = 0$ is often used. It can be shown that in such a case the beta distribution is defined if

$$\omega \leq 2V \tag{3.19}$$

where the coefficient of variation $V = \sigma / \mu$. For $\omega = 2V$ the curve becomes the type III Pearson distribution (see Section 3.3). Therefore if the input parameters are the mean μ, standard deviation σ and skewness $\omega \leq 2V$, the beta distribution with the lower limit at zero ($a = 0$) is fully described. The upper limit b of the beta distribution with the lower limit at zero follows from the relation (3.15)

$$b = \frac{\mu(c+d)}{c} = \frac{\mu(1+V(2+\omega V))}{(2V-\omega)} \tag{3.20}$$

In equation (3.20) the parameters c and d are substituted by the following expressions

$$c = -\frac{\omega}{2V}\frac{(2V-\omega)^2-(4+\omega^2)}{(V\omega+2)^2-(4+\omega^2)} \tag{3.21}$$

$$d = \frac{\omega}{2}\frac{(2V-\omega)^2-(4+\omega^2)}{(V\omega+2)^2-(4+\omega^2)}\frac{2+\omega V}{\omega-2V} \tag{3.22}$$

which follow from general equations (3.16) to (3.18) for $a = 0$.

Example 3.3.

Given the mean μ = 25 mm, standard deviation 10 mm (V = 0,40) and skewness ω = 0,5, assess the parameters of a beta distribution with the lower bound at zero (a = 0) for a reinforcement cover layer. Equation (3.19) is satisfied ($0{,}5 < 2 \times 0{,}4$). From equations (3.21) and (3.22) it follows that

$$c = -\frac{0{,}5}{2\times 0{,}4}\frac{(2\times 0{,}4-0{,}5)^2-(4+0{,}5^2)}{(0{,}4\times 0{,}5+2)^2-(4+0{,}5^2)} = 4{,}407$$

$$d = \frac{0{,}5}{2}\frac{(2\times 0{,}4-0{,}5)^2-(4+0{,}5^2)}{(0{,}4\times 0{,}5+2)^2-(4+0{,}5^2)}\frac{2+0{,}5\times 0{,}4}{0{,}5-2\times 0{,}4} = 12{,}927$$

The upper bound of the distribution b follows from equation (3.20) that

$$b = \frac{25\times(4{,}407+12{,}927)}{4{,}407} = 98{,}325$$

The beta distribution having the assessed parameters is shown in Figure 3.4 together with a corresponding normal, lognormal and Gamma distribution with the lower bound at zero and the same mean μ and standard deviation σ.

Figure 3.4 further shows that the normal distribution (skewness $\omega = 0$) leads to the occurrence of negative values, which may not correspond to the real conditions for the reinforcement cover layer. According to equation (3.11) the lognormal distribution with lower limit at zero has skewness $\omega = 1{,}264$, which does not correspond to experimental results and leads to an overestimation of the occurrence of positive deviations (which may further lead to unfavourable consequences in the reliability analysis of the reinforced concrete element).

The gamma distribution has, according to equation (3.13), a skewness $\omega = 2V = 0{,}8$, which is closer to the experimental value 0,5. The most convenient seems to be the beta distribution having the skewness $\omega = 0{,}5$ corresponding exactly to the experimental results.

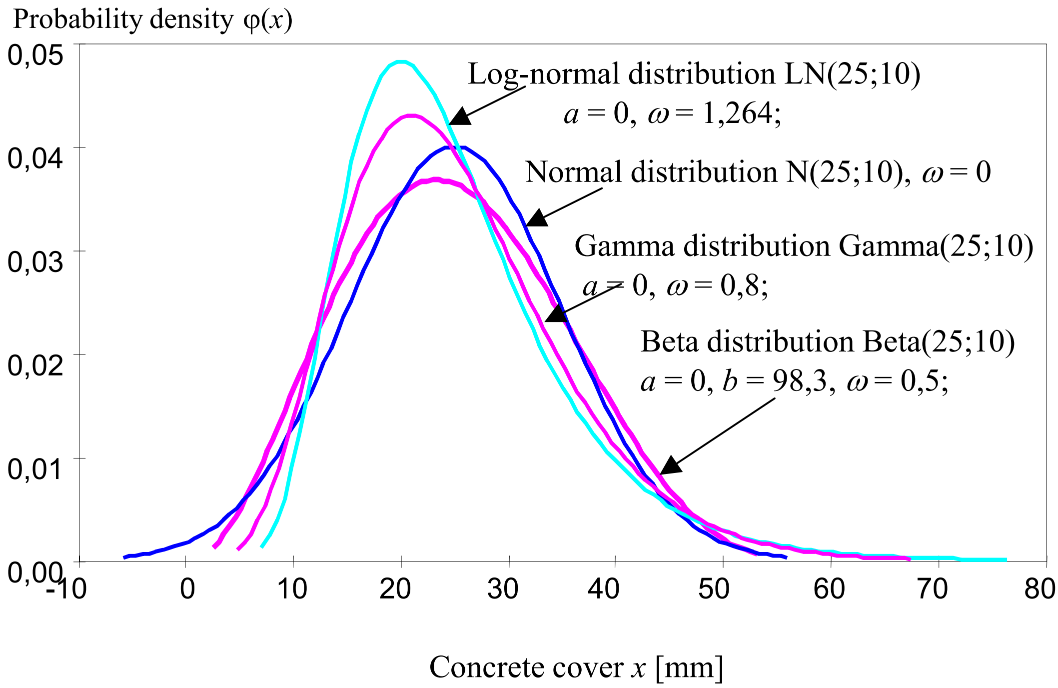


Figure 3.4. Normal, Lognormal, Gamma and Beta distributions for the concrete cover layer of reinforcement in a reinforced concrete element

It should be mentioned that mathematical statistics offers a number of “goodness of fit tests“ for evaluation of fitness of a distribution as a theoretical model for obtained experimental results (see for example documents [4, 12, 13] and a number recently developed ISO standards). The above discussion can therefore be supplemented by statistical tests. On the other hand it is essential to remark that goodness of fit tests very often fail and do not lead to an unambiguous result. In such a case the selection of a convenient model depends on the character of the basic variable, on available experience and on common experience.

3.5 Gumbel and other distributions of extreme values

The extreme values (maximal or minimal) in a population of a certain size are random variables and their distribution is very important in the theory of structural reliability. Three types of extreme values distribution denoted as types I, II and III are usually covered in the specialised literature. Each of the types has two versions – one for the distribution of minimal values, the second for maximal values distribution. All these types of distribution have a simple exponential shape and are convenient to work with. We will describe in detail the type I extreme value distribution, which is commonly called the Gumbel distribution. Description of other types of distribution can be found in textbook [12,13] and in specialised literature [15, 16, 17, 18, 19].

The distribution function for the type I maximal values distribution (Gumbel distribution of maximum values) has the form

$$\Phi(x) = \exp(-\exp(-c(x - x_{mod}))) \tag{3.23}$$

It is a distribution defined on an infinite interval, which depends on two parameters: on mode x_{mod} and parameter $c > 0$. By differentiating the distribution function we obtain the probability density function in the form

$$\varphi(x) = c \exp(-c\,(x - x_{mod}) - \exp(-c(x - x_{mod}))) \tag{3.24}$$

Both the parameters x_{mod}, c of the Gumbel distribution can be assessed from the mean μ and standard deviation σ

$$x_{mod} = \mu - 0{,}577\sqrt{\frac{6\sigma}{\pi}} \tag{3.25}$$

$$c = \frac{\pi}{\sqrt{6\sigma}} \tag{3.26}$$

Skewness and kurtosis of the distribution are constant: $\omega = 1{,}14$, $\varepsilon = 2{,}4$.

An important characteristic of the Gumbel distribution is the simple transformation of the distribution function $\Phi(x)$ of the original distribution to the distribution function $\Phi_N(x)$ describing the maxima of populations that are N times greater than the original population with mean μ and standard deviation σ. If the individual multiples of the original population are mutually independent, then it holds for the distribution function $\Phi_N(x)$

$$\Phi_N(x) = (\Phi(x))^N \tag{3.27}$$

By substitution of equation (3.23) into equation (3.27) we obtain the distribution function $\Phi_N(x)$ as

$$\Phi_N(x) = \exp(-\exp(-c(x - x_{mod} - \ln N/c))) \tag{3.28}$$

so the mean μ_N and standard deviation σ_N of maxima of populations that are N times greater than the original population are

$$\mu_N = \mu + \ln N/c = \mu + 0{,}78 \ln N\, \sigma,\ \sigma_N = \sigma \tag{3.29}$$

Thus the standard deviation σ_N of the greater population is equal to the standard deviation of the original population, $\sigma_N = \sigma$, but the mean μ_N is greater than the original value μ by 0.78 lnN/c.

Example 3.4.

One-year maxima of wind pressure are described by Gumbel distribution with a mean $\mu_1 = 0{,}35$ kN/m^2, $\sigma_1 = 0{,}06$ kN/m^2. The corresponding parameters of the 50-year maximum value distribution, i.e. parameters μ_{50} and σ_{50}, follow from equation (3.29)

$$\mu_{50} = 0{,}35 + 0{,}78 \times \ln(50 \times 0{,}06) = 0{,}53 \text{ kN/m}^2,\ \sigma_{50} = 0{,}06 \text{ kN/m}^2$$

Figure 3.5 shows both distributions of one-year and fifty-year maxima of wind pressure described by the Gumbel distribution.

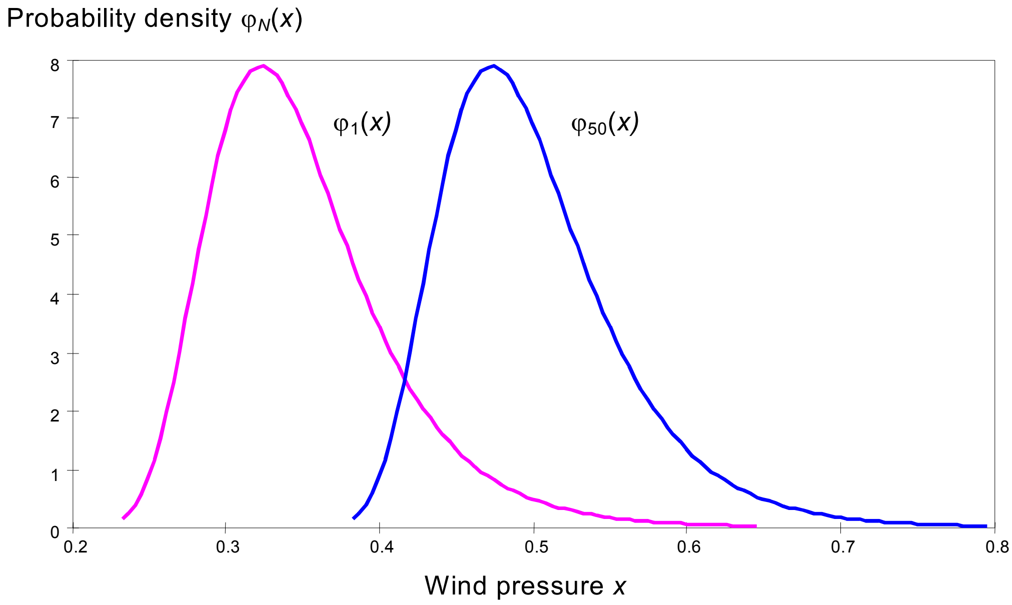


Figure 3.5. Distribution of maximum wind pressure over the periods of 1 year and 50 years.

The distribution function of type I minimal values distribution (Gumbel distribution of minimum values) has the form

$$\Phi(x) = 1 - \exp(-\exp(-c(x_{\text{mod}} - x))) \tag{3.30}$$

This distribution is symmetrical to the distribution of maximal values given by equation (3.23). It is therefore also defined on an open interval and depends on two parameters: on mode x_{mod} and parameter $c > 0$. By differentiating the distribution function we obtain the probability density function in the form

$$\varphi(x) = c \exp(-c\,(x_{\text{mod}} - x) - \exp(-c\,(x_{\text{mod}} - x))) \tag{3.31}$$

Both these parameters can be assessed from the mean μ and standard deviation σ

$$x_{\text{mod}} = \mu + 0{,}577\sqrt{\frac{6\sigma}{\pi}} \tag{3.32}$$

$$c = \frac{\pi}{\sqrt{6\sigma}} \tag{3.33}$$

The probability density function of the minimum values is symmetrical to the shape of maximal values relative to mode x_{mod}, as it is apparent from Figure 3.6.

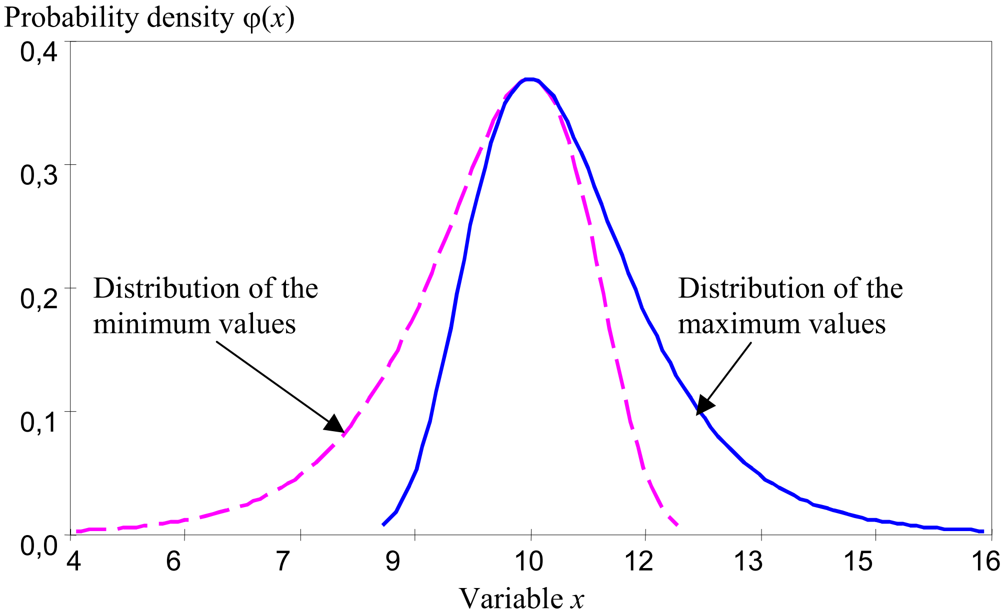


Figure 3.6. The Gumbel distribution of the minimum and maximum values.

In a similar way the type II distribution, the so-called Fréchet distribution, and type III distribution, the so-called Weibull distribution, are defined. All the three types of distribution complement each other with respect to the possible values of skewness ω. Each type covers a certain area of skewnesses, as indicated in Figure 3.7.

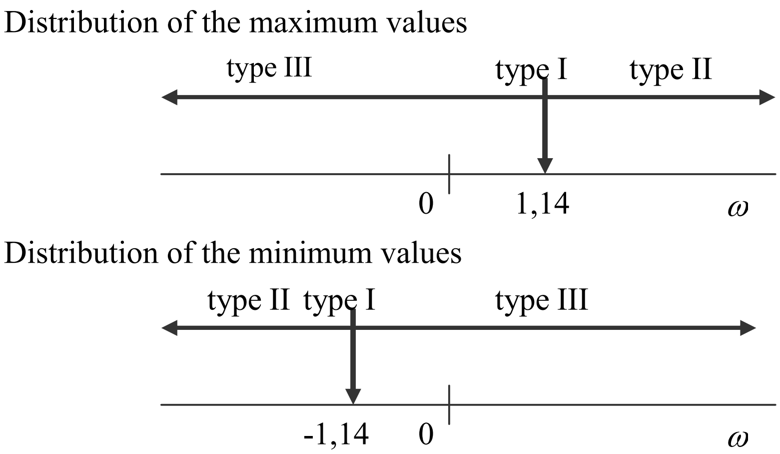


Figure 3.7. Types of distribution of extreme values versus the skewness ω.

The extreme values distributions of the type I and II are often used to describe random variables depending on the maximal values of populations (for example climatic actions). The type II is particularly convenient for variables with high skewness $\omega > 1,14$ (for example for flood discharge that have $\omega \sim 2$). The extreme values distribution of the type III is usually applied for random variables depending on the minimal values of populations (e.g. strength and other material properties) assuming that $\omega > -1,14$.

3.6 Function of random variables

In general many variables entering reliability analysis of structures may be considered as a function of basic variables $\boldsymbol{X} = [X_1, X_2, \dots , X_n]$. For example resistance R or load effect E may be given as a function

$$Z = \mathrm{F}(\boldsymbol{X}) \tag{3.34}$$

where $\boldsymbol{X} = [X_1, X_2, \dots , X_n]$ denotes a vector of basic variables. Then the resulting variable Z is a random variable and its characteristics may be derived from relevant characteristics of basic variables $\boldsymbol{X} = [X_1, X_2, \dots , X_n]$. Usually three moment parameters, the mean μ, standard deviation σ and skewness ω, are used for a first assessment of the resulting variable Z. Experience shows that using derived moment parameters (μ, σ and ω) three parameter lognormal distribution provides satisfactory approximation of Z. However, the software VaP [9] applies a more accurate approximation based on four moment parameters (μ, σ, ω and kurtosis ε).

Appendix 2 of this Chapter provides approximate expressions for fundamental functions of two basic variables that can be used in assessment of failure probability in case of small number of basic variables.

4 ESTIMATION OF FRACTILES

4.1 Fractile of a theoretical model

One of the most important keywords of the theory of structural reliability is the term fractile of a random variable X (or of its probability distribution), sometimes called also quantile. Appendix 3 to this Chapter provides a review of formulas for determining fractiles of most important theoretical models of continuous random variables.

Let us recall the definition of the fractile. For a given probability p, the p-fractile x_p denotes such a value of the random variable X, that values less than or equal to x_p occur just with the probability p. If $\Phi(x)$ is distribution function of the random variable X, then it follows from equation (2.6) that the value $\Phi(x_p)$ of the distribution function $\Phi(x)$ at the point x_p is equal to the probability p

$$\mathrm{P}(X \le x_\mathrm{p}) = \Phi(x_p) = p \tag{4.1}$$

The same definition holds also for standardised random variable U (given by transformation equation (3.2)) when in equation (4.1) U is substituted for X and u_p is substituted for x_p. Fractiles u_p of standardised random variables U are commonly available in tables. Figure 4.1 illustrates the definition of the fractile described by equation (4.1) for standardised random variable U; it shows distribution function $\Phi(u)$, probability density function $\varphi(u)$, probability p (approximately equal to 0,05) and fractile u_p for normal standardised distribution U.

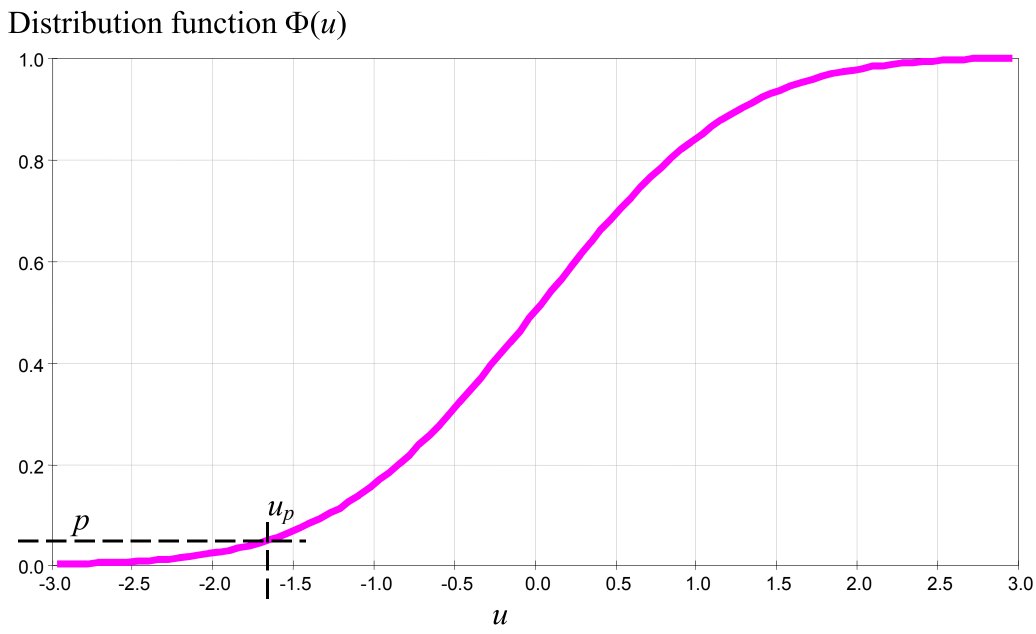
In general fractile x_p of the original random variable X may be calculated using tables for u_p available for standardised random variables U with a relevant type of distribution. It follows from transformation (3.2) that the fractile x_p may be determined from the fractile of the standardised random variable u_p (found in available tables) using relationship

$$x_p = \mu + u_p\sigma = \mu\,(1 + u_p\, V) \tag{4.2}$$

where μ denotes the mean, σ the standard deviation and V the coefficient of variation of the observed variable X.

If the probability $p < 0{,}5$, then the value x_p is called the lower fractile, for $p > 0{,}5$ the x_p is called the upper fractile. Figure 4.2 shows the lower and upper fractiles u_p of a standardized random variable U with normal distribution for probabilities p = 0,05 and 0,95, and thus denoted $u_{0,05}$ and $u_{0,95}$.

The fractile corresponding to the probability p = 0,05, is usually applied for an assessment of the characteristic value of material properties (strength of concrete, yield point of steel, masonry strength). However, the design values of dominant variables are fractiles, which correspond to a lower probability ($p \cong 0{,}001$), design values of variables which are not dominant are fractiles corresponding to a greater probability ($p \cong 0{,}10$).



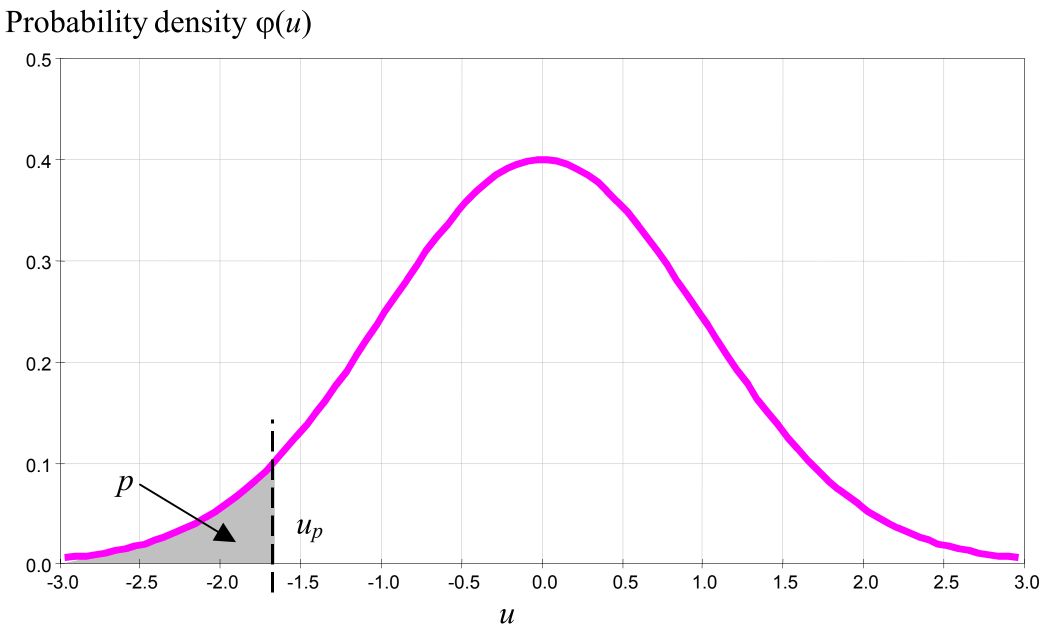


Figure 4.1. Definition of the fractile for the standardised random variable U.

The values u_p of the lower fractile of a standardized random variable U having normal distribution for selected probabilities p are given in Table 4.1. Considering the symmetry of the normal distribution, the values u_p of the upper fractile can be assessed from Table 4.1 by substituting of p by 1 - p and by changing the sign of values u_p (from negative to positive).

Detailed tables can be found e.g. in books [12, 13], in the International Standard ISO 12491 [4] and in specialised literature [18, 19, 20].

For a standardized random variable with a general three-parametric lognormal distribution the value u_p of the standardized random variable is dependent on skewness ω. The values u_p for selected skewnesses ω and probabilities p are given in Table 4.2.

Table 4.1. Fractile u_p of a standardized random variable with normal distribution

p	10^{-7}	10^{-6}	10^{-5}	10^{-4}	0,001	0,010	0,050	0,100	0,200	0,500
u_p	-5,199	-4,753	-4,265	-3,719	-3,091	-2,327	-1,645	-1,282	-0,841	0,000

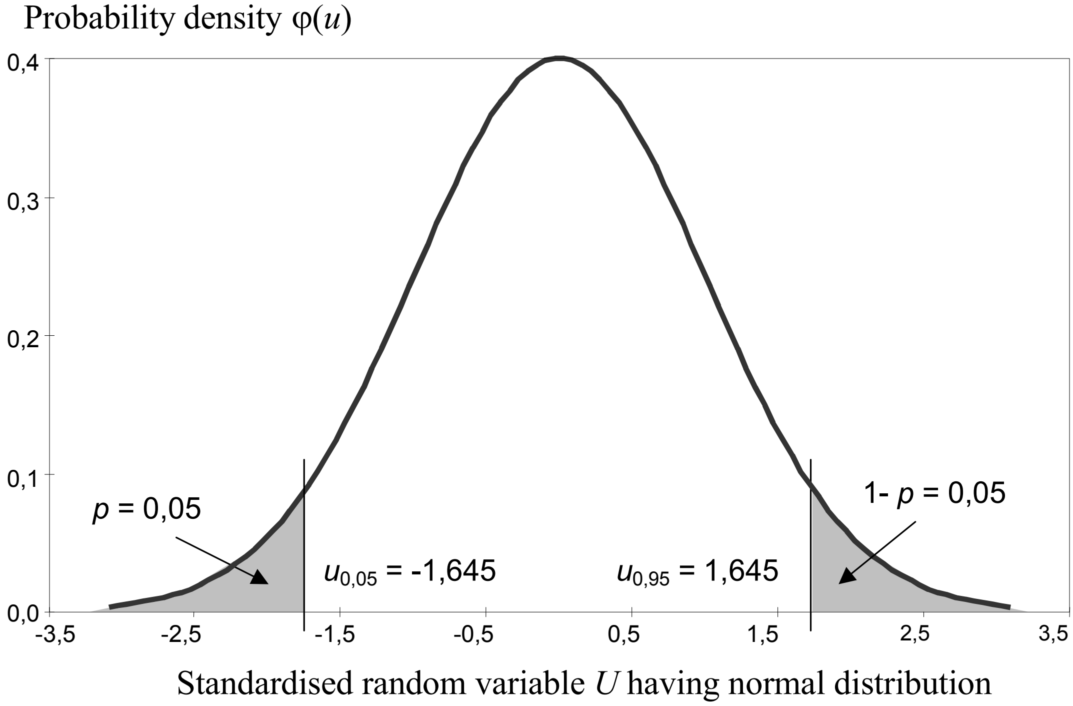


Figure 4.2. The lower and upper fractiles of a standardized random variable U having normal distribution

In the case of a lognormal distribution with lower limit at zero, which is described in section 3.2, it is possible to calculate the fractile from the value of fractile of a standardized random variable with normal distribution using the relation

$$x_p = \frac{\mu}{\sqrt{1+V^2}} \exp\left(u_{norm,p}\sqrt{\ln(1+V^2)}\right) \tag{4.3}$$

where $u_{\mathrm{norm,p}}$ is the fractile of a standardized random variable with normal distribution, μ is the mean and V the coefficient of variation of the variable X. An approximation of relation (4.3) is often applied in the form

$$x_p \cong \mu \exp\,(u_{\mathrm{norm,p}} \times V) \tag{4.4}$$

whose accuracy is fully satisfactory for the coefficient of variations $V < 0{,}2$, but it is commonly used for greater V as well.

Table 4.2. Fractile u_p of a standardized random variable having lognormal distribution

ω	Probability p												
	10^{-4}	10^{-3}	0,01	0,05	0,10	0,20	0,50	0,80	0,90	0,95	0,99	$1\text{-}10^{-3}$	$1\text{-}10^{-4}$
-2,0	-9,52	-6,24	-3,52	-1,89	-1,24	-0,61	0,24	0,77	0,97	1,89	1,28	1,42	1,49
-1,5	-7,97	-5,51	-3,31	-1,89	-1,29	-0,68	0,20	0,81	1,04	1,21	1,45	1,65	1,77
-1,0	-6,40	-4,70	-3,03	-1,85	-1,32	-0,74	0,15	0,84	1,13	1,34	1,68	1,99	2,19
-0,5	-4,94	-3,86	-2,70	-1,77	-1,32	-0,80	0,08	0,85	1,21	1,49	1,98	2,46	2,81
0,0	-3,72	-3,09	-2,33	-1,65	-1,28	-0,84	0,00	0,84	1,28	1,65	2,33	3,09	3,72
0,5	-2,81	-2,46	-1,98	-1,49	-1,21	-0,85	-0,08	0,80	1,32	1,77	2,70	3,86	4,94
1,0	-2,19	-1,99	-1,68	-1,34	-1,13	-0,84	-0,15	0.74	1,32	1,85	3,03	4,70	6,40
1,5	-1,77	-1,65	-1,45	-1,21	-1,04	-0,81	-0,20	0,68	1,29	1,89	3,31	5,51	7,97
2,0	-1,49	-1,42	-1,28	-1,89	-0,97	-0,77	-0,24	0,61	1,24	1,89	3,52	6,24	9,52

Example 4.1.

Let us assess the fractile x_p of a normal and lognormal distribution with lower limit at zero for p = 0,001; 0,01; 0,05 and 0,10, if V = 0,3. We know that the lognormal distribution with lower limit at zero has, in this case, a positive skewness ω = 0,927 (according to equation (3.11)), which needs to be known for interpolation in Table 4.2. The resultant values x_p are given in the following table in the form of dimensionless coefficients x_p/μ (expressing the ratio of the fractile to the mean), which were for normal and for lognormal distribution assessed by different ways.

Table of coefficients x_p/μ.

Coefficient x_p/μ for	Probability p			
	0,001	0,010	0,050	0,100
normal distribution, equation (4.2) and Table 4.1	0,073	0,302	0,506	0,615
lognormal distribution, equation (4.2) and Table 4.2	0,385	0,483	0,591	0,658
lognormal distribution, equation (4.3) and Table 4.1	0,387	0,484	0,591	0,657
lognormal distribution, equation (4.4) and Table 4.1	0,396	0,496	0,610	0,681

Table of coefficients x_p/μ shows the expected difference between the fractiles of normal and of lognormal distributions. The lower fractile of normal distribution is significantly lower than the corresponding fractile of lognormal distribution particularly for small probabilities p. The table also shows that the approximate formula (4.4) provides satisfactory results for computation of fractile of lognormal distribution (the error will decrease with decreasing coefficient of variation V).

The fractile of gamma distribution can be calculated from the available tables for type III Pearson distribution [12, 13]. To calculate the fractile of beta distribution, the available tables of incomplete beta function may be used or it can be assessed by integration of probability density function according to definition (4.1). However, when it is needed (and neither appropriate tables nor software product are available), the fractile of beta distribution, which is bell shaped (for shape parameters it holds that $c > 2$ and $d > 2$), may be assessed approximately from equation (4.2) using table values of u_p for a standardized lognormal distribution, having the same skewness ω as the beta distribution. Analogical procedure may be used for other types of distribution, too.

The fractile x_p can be easily assessed for Gumbel distribution. From equation (3.23) and definition (4.1) follows an explicit relation for x_p directly dependent on the probability p

$$x_p = x_{\mathrm{mod}} - \frac{1}{c}\ln(-\ln(p)) \cong \mu - (0{,}45 + 0{,}78\ln(-\ln(p)))\sigma \tag{4.5}$$

where mode x_{mod} and parameter c are substituted by relations (3.25) and (3.26).

Example 4.2.

Let us determine the upper fractile of wind pressure from Example 3.4 described by Gumbel distribution when probability $p = 0{,}98$ is considered. It is known from Example 3.5 that for the one-year maximum $\mu_1 = 0{,}35$ kN/m^2, $\sigma_1 = 0{,}06$ kN/m^2. The fractile $x_{0,98}$ for such parameters follows from equation (4.5)

$$x_{0,98} = 0{,}35 - (0{,}45 + 0{,}78 \times \ln(-\ln(0{,}98))) \times 0{,}06 = 0{,}51 \text{ kN/m2}$$

The corresponding fractile of the maximum for a period of 50 years (as shown in Example 3.4 that $\mu_{50} = 0{,}53$ kN/m^2, $\sigma_{50} = 0{,}06$ kN/m^2) is

$$x_{0,98} = 0{,}53 - (0{,}45 + 0{,}78 \times \ln(-\ln(0{,}98))) \times 0{,}06 = 0{,}69 \text{ kN/m}^2$$

Simple mathematical procedures, including the computation of fractile, are some of the reasons of the wide popularity of Gumbel, distribution which is frequently used for random variables describing climatic and other variable actions that are defined by maximal values for a given period (e.g. during one year).

However, theoretical models are not always known in practical applications. In civil engineering, the fractile of a random variable (e.g. strength of a new or unknown material) has to be assessed from a limited sample, the size n of which may be very small (sometimes less than 10). Furthermore, considered random variables may have a high variability (the coefficient of variation is sometimes greater than 0,30). Assessment of the fractile of a population from a sample is then a serious problem, which is in mathematical statistics solved by various methods of estimation theory. In the following three basic methods are shortly described: the coverage method, the prediction method and the Bayesian method for estimation of the population fractile.

4.2 Coverage method of fractile estimation

The keyword of the coverage method for the fractile estimation from a sample of a limited size n is the confidence γ, i.e. the probability (usually 0,75, 0,90 or 0,95) that the estimated value covers the population fractile (that is why the method is called coverage method). The estimator $x_{p,\mathrm{cover}}$ of the lower fractile x_p is determined by the coverage method in such a way that

$$\mathrm{P}(x_{\mathrm{p,cover}} < x_{\mathrm{p}}) = \gamma \tag{4.6}$$

Thus, the estimator $x_{p,\mathrm{cover}}$ is lower (on the safe side of the lower fractile) than the unknown fractile x_p with the probability (confidence) γ.

In the following summary practical formulas are given without being derived, assuming that the population has a general three-parameter distribution characterized by skewness ω, known from previous experience. In addition it is assumed that the mean μ of the population is never known in advance and the estimation is based by the average m obtained from a sample. The standard deviation σ of the population is assumed to be either known (and then it is used) or unknown (and then the sample standard deviation s or the coefficient of variation V is used instead of σ).

If the standard deviation σ of the population is known from previous experience, the estimator $x_{p,\mathrm{cover}}$ of the lower p-fractile is given as

$$x_{\mathrm{p,cover}} = m - \kappa_{\mathrm{p}}\ \sigma \tag{4.7}$$

If the standard deviation of the population σ is unknown, then the sample standard deviation s is considered

$$x_{\mathrm{p,cover}} = m - k_{\mathrm{p}}\ s \tag{4.8}$$

Coefficients of estimation $\kappa_p = \kappa\ (\omega, p, \gamma, n)$ and $k_p = k\ (\omega, p, \gamma, n)$ depend on skewness ω, on probability p corresponding to the fractile x_p that is estimated, on confidence γ and on the size n of the population. The knowledge of confidence γ that the estimator $x_{p,\mathrm{cover}}$ will be on the safe side of the real value is the greatest advantage of the classic coverage method. In documents [1, 2] the confidence γ is recommended by the value 0,75. In the cases of increased reliability demands when a detailed reliability analysis is required, a higher value of confidence, say of 0,95, may be more appropriate [4].

4.3 Prediction method of fractile estimation

According to the prediction method [4] the lower p-fractile x_p is estimated by the so-called prediction limit $x_{p,\mathrm{pred}}$ for which it holds that a new value x_{n+1} randomly drawn from the population will be lower than the estimator $x_{p,\mathrm{pred}}$ only with the probability p, i.e. it holds that

$$\mathrm{P}(x_{n+1} < x_{p,\mathrm{pred}}) = p \tag{4.9}$$

It can be shown that for growing n the estimator $x_{p,\mathrm{pred}}$ defined in this way is asymptotically approaching the unknown fractile x_p. It can be also shown that the estimator $x_{p,\mathrm{pred}}$ corresponds approximately to the estimator obtained by the coverage method $x_{p,\mathrm{cover}}$ for confidence $\gamma = 0,75$ [4].

If the standard deviation σ of the population is known, then the lower p-fractile is estimated by the value $x_{p,\mathrm{pred}}$ according to the relation

$$x_{p,\mathrm{pred}} = m + u_p\ (1/n + 1)^{1/2}\ \sigma \tag{4.10}$$

where $u_p = u(\omega, p)$ is the p-fractile of a standardized lognormal distribution having the skewness ω. If the distribution of the variable X is normal then u_p is the p-fractile of standardised normal distribution.

If, however, the standard deviation of population is unknown, then the sample standard deviation s must be considered instead of σ

$$x_{p,\mathrm{pred}} = m + t_p\ (1/n + 1)^{1/2}\ s \tag{4.11}$$

where $t_p = t(\omega, p, \nu)$ is the p-fractile of the generalized Student's t-distribution for $\nu = n - 1$ degrees of freedom, which has a skewness ω (information about Student‘s distribution and about the number of degrees of freedom may be obtained from the textbook [12,13] and from other specialised sources [18, 19]). If the distribution of the variable X is normal, then u_p is the p-fractile of standardised normal distribution Student's t-distribution for $\nu = n - 1$ degrees of freedom.

4.4 Coefficients of the coverage and prediction methods

The coverage and prediction methods represent two basic procedures of estimation of the population's fractile from the available sample of a limited size n. If the standard deviation of the population σ is known, then equations (4.7) and (4.10) are applied in which two analogical coefficients κ_p and $-u_p(1/n + 1)^{1/2}$ appear. Both of these coefficients depend on the sample size n, coefficient κ_p of the coverage method depends furthermore on the confidence γ. Table 4.3 shows the coefficients κ_p and $-u_p(1/n + 1)^{1/2}$ for $p = 0{,}05$ and selected values of n and γ when normal distribution of the population is assumed.

Table 4.3. Coefficients κ_p and $-u_p(1/n + 1)^{1/2}$ from equations (4.7) and (4.10) for $p = 0{,}05$ and normal distribution of the population (when σ is known).

Coefficient		Sample size n								
		3	4	5	6	8	10	20	30	∞
κ_p	$\gamma = 0{,}75$	2,03	1,98	1,95	1,92	1,88	1,86	1,79	1,77	1,64
	$\gamma = 0{,}90$	2,39	2,29	2,22	2,17	2,10	2,05	1,93	1,88	1,64
	$\gamma = 0{,}95$	2,60	2,47	2,38	2,32	2,23	2,17	2,01	1,95	1,64
$-u_p(1/n+1)^{1/2}$		1,89	1,83	1,80	1,77	1,74	1,72	1,68	1,67	1,64

It is evident from Table 4.3 that with the growing sample size n both the coefficients approach the value 1,64, which holds for a theoretical model of the normal distribution (see Table 4.1). The coefficient κ_p of the coverage method increases with increasing confidence γ. Note that for confidence $\gamma = 0{,}75$ it holds that $\kappa_p \cong -u_p(1/n + 1)^{1/2}$. Thus, for $\gamma = 0{,}75$ the coverage method leads approximately to the same estimator as the prediction method, $x_{p,\text{cover}} \cong x_{p,\text{pred}}$ (for greater confidence $\gamma > 0{,}75$ the $x_{p,\text{cover}} < x_{p,\text{pred}}$).

If the standard deviation of the population σ is unknown, equations (4.8) and (4.11) are applied in which two analogical coefficients k_p and $-t_p(1/n + 1)^{1/2}$ appear. Both of these coefficients depend again on the sample size n, coefficient k_p of the coverage method depends furthermore on the confidence γ. Table 4.4 and Figure 4.3 show the values of coefficients k_p and $-t_p(1/n + 1)^{1/2}$ for $p = 0{,}05$ and selected values of n and γ when normal distribution of the population is assumed.

Table 4.4. Coefficients k_p and $-t_p(1/n + 1)^{1/2}$ from equations (4.8) and (4.11) for $p = 0{,}05$ and normal distribution of the population (when σ is unknown).

Coefficient		Sample size n								
		3	4	5	6	8	10	20	30	∞
k_p	$\gamma = 0{,}75$	3,15	2,68	2,46	2,34	2,19	2,10	1,93	1,87	1,64
	$\gamma = 0{,}90$	5,31	3,96	3,40	3,09	2,75	2,57	2,21	2,08	1,64
	$\gamma = 0{,}95$	7,66	5,14	4,20	3,71	3,19	2,91	2,40	2,22	1,64
$-t_p(1/n+1)^{1/2}$		3,37	2,63	2,33	2,18	2,00	1,92	1,76	1,73	1,64

It is obvious from Table 4.4 and Figure 4.3 that with increasing sample size n both the coefficients k_p and $-t_p(1/n + 1)^{1/2}$ approach the value 1,64, which is valid for a theoretical model of the normal distribution (see Table 4.1). In case of the coverage method, the coefficient k_p increases with increasing confidence γ and the relevant estimators $x_{p,\text{cover}}$ of the lower fractile are decreases (on the safe side). Note, that as in the case of known standard deviation σ both coefficients are approximately equal, $k_p \cong -t_p(1/n + 1)^{1/2}$ and for confidence γ

= 0,75 the coverage method leads to approximately the same estimator, $x_{p,\text{cover}} \cong x_{p,\text{pred}}$, as the prediction method.

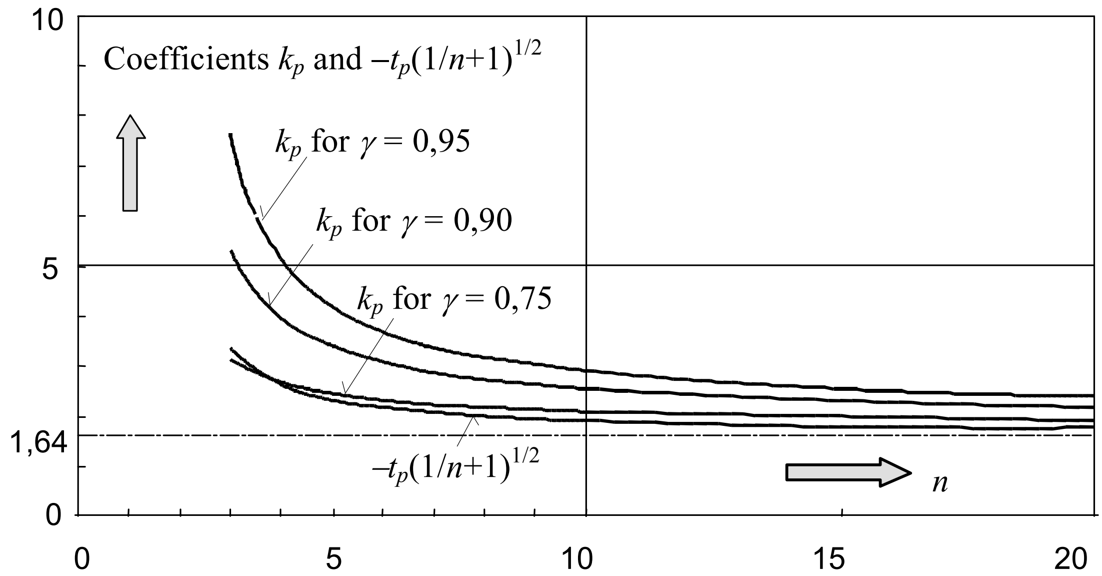


Figure 4.3. Coefficients k_p and $-t_p(1/n + 1)^{1/2}$ for p = 0,05 and normal distribution of the population (when σ is unknown).

Also the skewness (asymmetry) of the population ω may affect significantly the estimator of the population's fractile. Tables 4.5 and 4.6 show the coefficients k_p from equation (4.8) for three value of the skewness $\omega = -1{,}0$, 0,0 and 1,0, probability p = 0,05 and confidence γ = 0,75 (Table 4.5) and γ = 0,95 (Table 4.6). Values of the coefficients from Table 4.6 are shown in Figure 4.4.

Table 4.5. Coefficient k_p from equation (4.8) for p = 0,05, γ = 0,75 and lognormal distribution having skewness ω (when σ is not known).

	Sample size n								
Skewness	3	4	5	6	8	10	20	30	∞
$\omega = -1{,}00$	4,31	3,58	3,22	3,00	2,76	2,63	2,33	2,23	1,85
$\omega = 0{,}00$	3,15	2,68	2,46	2,34	2,19	2,10	1,93	1,87	1,64
$\omega = 1{,}00$	2,46	2,12	1,95	1,86	1,75	1,68	1,56	1,51	1,34

Table 4.6. Coefficient k_p from equation (4.8) for p = 0,05, γ = 0,95 and lognormal distribution having the skewness ω (when σ is not known).

	Sample size n								
Skewness	3	4	5	6	8	10	20	30	□
$\omega = -1{,}00$	10,9	7,00	5,83	5,03	4,32	3,73	3,05	2,79	1,85
$\omega = 0{,}00$	7,66	5,14	4,20	3,71	3,19	2,91	2,40	2,22	1,64
$\omega = 1{,}00$	5,88	3,91	3,18	2,82	2,44	2,25	1,88	1,77	1,34

It is evident from Tables 4.5 and 4.6 that as the sample size n increases, the coefficients k_p approach the values of u_p, which are valid for theoretical model of lognormal distribution (see Table 4.2). Thus, the influence of the skewness ω does not disappear when n

→ ∞, and it is especially significant for small samples and greater confidence $\gamma = 0{,}95$ (see Figure 4.4).

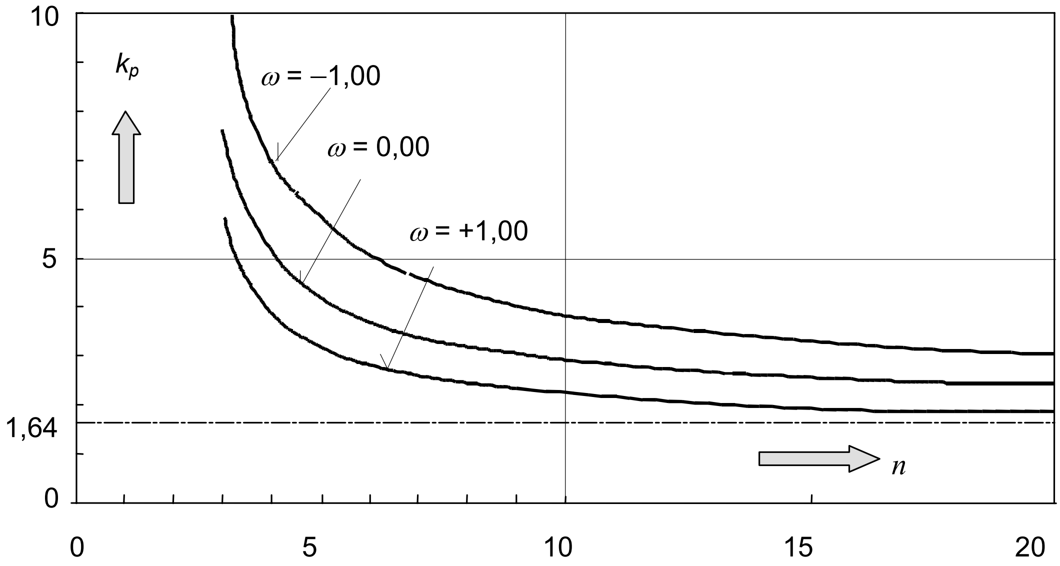


Figure 4.4. Coefficient k_p for $p = 0{,}05$ and confidence $\gamma = 0{,}95$ (when σ is unknown).

A similar dependence on skewness may be observed in the case of the generalized Student‘s t-distribution for which the fractiles t_p are given in Table 4.7. These values t_p are applied in the prediction method using formula (4.11) and further in the Bayes‘ method. That is why Table 4.7 gives directly the values of fractiles t_p depending on the number of degrees of freedom ν. Similarly as in Tables 4.5 and 4.6 the probability $p = 0{,}05$ and three skewnesses $\omega = -1{,}0$; 0,0 and 1,0 are considered.

Table 4.7. Coefficient $-t_p$ from equation (4.11) for $p = 0{,}05$ and lognormal distribution with skewness ω (when σ is unknown).

	Coefficient $-t_p$ for $\nu = n - 1$ degrees of freedom								
Skewness	3	4	5	6	8	10	20	30	∞
$\omega = -1{,}00$	2,65	2,40	2,27	2,19	2,19	2,04	1,94	1,91	1,85
$\omega = 0{,}00$	2,35	2,13	2,02	1,94	1,86	1,81	1,72	1,70	1,64
$\omega = 1{,}00$	1,92	1,74	1,64	1,59	1,52	1,48	1,41	1,38	1,34

It follows from Table 4.7 that as the size of the sample n increases, the values of t_p approach the theoretical values of u_p, which are valid for a model of lognormal distribution with the appropriate skewness and are given in Table 4.2. Therefore, the influence of the skewness again (as in the case of k_p) does not disappear for $n \to \infty$, but it is especially significant for small samples (it increases with decreasing sample size n).

Example 4.3.

A sample of the size $n = 5$ measurements of strength of concrete has the mean m = 29,2 MPa and the standard deviation s = 4,6 MPa. We assume that the population is normal and that its standard deviation σ is unknown. The characteristic strength $f_{ck} = x_p$, for $p = 0{,}05$ is firstly assessed by the coverage method. If the confidence is $\gamma = 0{,}75$, then it follows from equation (4.8) and Table 4.4 that

$$x_{p,\text{cover}} = 29{,}2 - 2{,}46 \times 4{,}6 = 17{,}9 \text{ MPa}$$

If the higher confidence $\gamma = 0{,}95$ is required, then

$$x_{p,\text{cover}} = 29{,}2 - 4{,}20 \times 4{,}6 = 9{,}9 \text{ MPa}$$

If the prediction method is used, then it follows from equation (4.11) and Table 4.4 that

$$x_{p,\text{pred}} = 29{,}2 - 2{,}33 \times 4{,}6 = 18{,}5 \text{ MPa}$$

The characteristic strength obtained by the prediction method is only a little greater than the value according to the coverage method with confidence $\gamma = 0{,}75$. However, if a higher confidence $\gamma = 0{,}95$ is required, then the prediction method leads to a value which is almost twice greater than the value obtained by the coverage method.

If the sample comes from a population with lognormal distribution and a positive skewness $\omega = 1$, then the coverage method with the confidence $\gamma = 0{,}75$ (Table 4.5) gives an estimator

$$x_{p,\text{cover}} = 29{,}2 - 1{,}95 \times 4{,}6 = 20{,}2 \text{ MPa}$$

which is a value by 13% greater than when the skewness is zero.

Similarly it follows for the prediction method from equation (4.11) and Table 4.7 that

$$x_{p,\text{pred}} = 29{,}2 - 1{,}74 \times \sqrt{\frac{1}{5} + 1} \times 4{,}6 = 20{,}4 \text{ MPa}$$

where the value t_p = -1,74 is given in Table 4.7 for $\omega = 1{,}0$ and $\nu = 5 - 1 = 4$. The resulting strength is in this case by 10% greater than the value, which corresponds to the normal distribution ($\omega = 0$).

4.5 Bayes' method of fractile estimation

If previous experience is available for a random variable (e.g. in the case of a long term production) it is possible to use so-called Bayes` method, which generally follows the idea of updating of probabilities described in section 2.5. The Bayes' method of fractile estimation is described here without deriving any important relations. More detailed description is given in documents ISO [3, 4] and other specialised literature [12, 13].

Assume that a sample of size n with an average m and standard deviation s is available. Note that degrees of freedom $\nu = n - 1$. Besides an average m' and sample standard deviation s' assessed from an unknown sample (of an unknown size n' and degrees of freedom ν') are available from previous experience. It is, however, assumed that both the samples come from the same population having the mean μ and the standard deviation σ. If this important assumption is valid, then the two samples may be combined. This could be a simple task if the individual values of the previous set were known, but that is not the case. However, the Bayes` method must be used.

Parameters of the combined sample are generally given by relations [3, 4]

$$n'' = n + n'$$

$$\nu'' = \nu + \nu' - 1 \text{ if } n' \geq 1,\ \nu'' = \nu + \nu' \text{ if } n' = 0 \tag{4.12}$$

$$m'' = (mn + m'n') / n''$$

$$s''^2 = (\nu s^2 + \nu' s'^2 + nm^2 + n'm'^2 - n''m''^2) / \nu''$$

The unknown sample size n' may be assessed using the relations for coefficients of variation of the mean and standard deviation $V(\mu)$ and $V(\sigma)$, (parameters μ and σ are considered as random variables in the Bayes‘ concept) for which it holds [3, 4]

$$n' = [s' / (m' V(\mu))]^2,\ \nu' = 1 / (2\ V(\sigma)^2) \tag{4.13}$$

Both the unknown variables n' and ν' may be assessed independently (generally $\nu' \neq n'$ - 1), depending on previous experience with the degree of uncertainty of estimator of the mean μ and standard deviation σ of the population.

The next step of the procedure applies the prediction method of fractile estimation. The Bayes‘ estimator $x_{p,\text{Bayes}}$ of the fractile is given by relationship similar to equation (4.11) for prediction estimator, assuming that the standard deviation σ of the population is not known

$$x_{p,\text{Bayes}} = m''+t''_p(1/n''+1)^{1/2} s'' \tag{4.14}$$

where $t''_p = t''_p(\omega, p, \nu'')$ is a fractile of the generalised Student‘s t-distribution having an appropriate skewness ω for ν'' degrees of freedom (that is generally different from the value n''– 1).

If the Bayes‘ method is applied for an assessment of material strength, the advantage may be taken of the fact that the long-term variability is constant. Then the uncertainty of an assessment of σ and the value $v(\sigma)$ are relatively small, variables ν' assessed according to equation (4.13) and ν'' assessed according to equation (4.12) are relatively high. This factor may lead to a favourable decrease of the value t''_p and to augmentation of the estimator of the lower fractile of x_p according to equation (4.14). On the other hand, uncertainties in assessment of the mean μ and the variable $v(\mu)$ are usually great and previous information may not affect significantly the resulting values n'' and m''.

If no previous information is available, then $n' = \nu' = 0$ and the resulting characteristics m'', n'', s'', ν'' equal the sample characteristics m, n, s, ν. In this case the Bayes‘ method is reduced to the prediction method and equation (4.14) becomes equation (4.11); if σ is known equation (4.10) is used. This particular form of the Bayes‘ method, when no previous information is available, is considered in Eurocode EN 1990 [1] and international standards ISO [2, 3].

Example 4.4.

If previous experience was available for Example 4.3, the Bayes‘ method could be used. Suppose that the information is m' = 30,1 MPa, $V(\mu)$ = 0,50, s' = 4,4 MPa, $V(\sigma)$ = 0,28. It follows from equation (4.13) that

$$n' = \left(\frac{4{,}4}{30{,}1}\frac{1}{0{,}50}\right)^2 < 1\ ,\nu' = \frac{1}{2\times 0{,}28^2} \approx 6$$

Further on these values are thus considered: n' = 0 and ν' = 6. Because $\nu = n - 1 = 4$, it follows from equation (4.12)

$$n'' = 5,\ \nu'' = 10,\ m'' = 29{,}2\ \text{MPa},\ s'' = 4{,}5\ \text{MPa}.$$

From equation (4.14) the fractile estimate follows as

$$x_{p,Bayes} = 29{,}2 - 1{,}81\times\sqrt{\frac{1}{5}+1}\times 4{,}5 = 20{,}3\ \text{MPa}$$

where the value $t_p'' = 1{,}81$ is given in Table 4.7 for $\omega = 0$ and $\nu'' = 10$. The resulting strength is thus greater (by 10%) than the value obtained by the prediction method.

If the population has lognormal distribution with skewness $\omega = 1$, then it follows from equation (4.14) considering the value $t_p'' = 1{,}48$ given in Table 4.7 that

$$x_{p,Bayes} = 29{,}2 - 1{,}48 \times \sqrt{\frac{1}{5} + 1} \times 4{,}5 = 21{,}9 \text{ MPa}$$

which is a value by 8% greater than the Bayes‘ estimator for $\omega = 0$.

Examples 4.3 and 4.4 clearly showed that the estimator of characteristic strength (fractile with probability $p = 0{,}05$) assessed from one sample may be expected within a broad range (in Examples 4.3 and 4.4 from 9,9 MPa to 21,9 MPa), depending on the applied method, required confidence, previous information and on assumptions concerning the population. Besides the alternatives considered in Examples 4.3 and 4.4, knowledge of the standard deviation σ of the population and assumption of the negative skewness (in the case of some materials of high strength) may be applied as well.

Even more significant differences in the resulting values may occur when design values of strength are being estimated, i.e. when fractiles corresponding to a small probability ($p \cong 0{,}001$) are considered. However, a direct estimation of such fractiles from a limited sample of the population is recommended only in such cases when a sufficient amount of information on the relevant random variable is available. In such cases, it is necessary to proceed carefully and, if possible, in co-operation with experts in the field of mathematical statistics.

4.6 Estimation of fractiles according to Eurocodes

Eurocode EN 1990 [1] gives in tables the coefficients for estimation of a fractile of a random variable with normal distribution (asymmetric distributions thus are not considered for the fractile estimation) from a sample for three probabilities $p = 0{,}05$ (for characteristic value x_k), $p = 0{,}001$ (for design value x_d of the dominant variable) and for $p = 0{,}10$ (for design value x_d of the non-dominating variable). As already mentioned above, the characteristic values x_k and design values x_d are defined as fractiles x_p, which correspond to a given probability p (application of these variables in structural design is explained in the following chapters).

For characteristic values of material properties a fractile corresponding to probability $p = 0{,}05$ is usually considered (however, for variables which describe variable loads the probability p is usually less than that), i.e. it holds

$$P(X < x_k) = 0{,}05 \tag{4.15}$$

For design values x_d of dominating variables it holds approximately that $p = 0{,}001$ (or another value close to this one), i.e. it holds

$$P(X < x_d) = 0{,}001 \tag{4.16}$$

Finally, for design values x_d of non-dominant variables it holds approximately that $p = 0{,}1$, i.e. it holds that

$$P(X < x_d) = 0{,}1 \tag{4.17}$$

A more detailed description of the dominating and non-dominating variables is given in Handbook 1.

The following Tables 4.8 and 4.10, which give the required coefficients for estimation of variables x_k and x_d according to equations (4.15) to (4.17), are adopted from the document [1] in its original version, even though the first Table 4.8 partially overlaps with the precedent Tables 4.3 and 4.4. Tables 4.8 and 4.9 are taken from the final version of EN 1990 [1]. Let us remark that all the coefficients in [1] are denoted by the symbol k_n, which is used also in the following tables.

Table 4.8. Coefficients k_n for a 5% characteristic value (see Tables 4.4 and 4.3).

	Sample size n										
Coefficient	1	2	3	4	5	6	8	10	20	30	∞
$-u_p(1/n+1)^{1/2}$, σ known	2,31	2,01	1,89	1,83	1,80	1,77	1,74	1,72	1,68	1,67	1,64
$-t_p(1/n+1)^{1/2}$, σ unknown	-	-	3,37	2,63	2,33	2,18	2,00	1,92	1,76	1,73	1,64

Table 4.9. Coefficients k_n for a design value x_d of a dominating variable, $P(X < x_d) = 0{,}001$.

	Sample size n										
Coefficient	1	2	3	4	5	6	8	10	20	30	∞
$-u_p(1/n+1)^{1/2}$, σ known	4,36	3,77	3,56	3,44	3,37	3,33	3,27	3,23	3,16	3,13	3,09
$-t_p(1/n+1)^{1/2}$, σ unknown	-	-	-	11,4	7,85	6,36	5,07	4,51	3,64	3,44	3,09

Table 4.10. Coefficients k_n for a design value x_d of a non-dominating variable, $P(X < x_d) = 0{,}1$.

	Sample size n										
Coefficient	1	2	3	4	5	6	8	10	20	30	∞
$-u_p(1/n+1)^{1/2}$, σ known	1,81	1,57	1,48	1,43	1,40	1,38	1,36	1,34	1,31	1,30	1,28
$-t_p(1/n+1)^{1/2}$, σ unknown	-	3,77	2,18	1,83	1,68	1,56	1,51	1,45	1,36	1,33	1,28

The assumption concerning knowledge of the standard deviation σ is replaced (inaccurately) in the document by the assumption that the coefficient of variation V is known. The original version of Table 4.9 [1] gives for the sample size of ∞ a wrong value of 3,04 for the coefficients (correct is 3,09). Let us also note that when knowledge of the standard deviation σ is assumed, Tables 4.8 to 4.10 give values of coefficients already for the sample size n = 1. Application of these values is, however, associated with significant statistical uncertainties and therefore a minimum sample size n = 3 is recommended here. Note, that Table 4.10 (for 0,1 fractile) is included only in the prestandard ENV 1991-1 and not in the final document EN 1990 [1].

Statistical methods for determining the characteristic and design values of resistance variables are provided in Annex D "Design assisted by testing" of EN 1990 [1]. Relevant basic variables describing structural resistance are described by lognormal distribution. The whole procedure is described in detail in the Annex D. Attached MATHCAD sheet "Mod_est.mcd" can be used to evaluate experimental data using the whole procedure. It is provided with explanatory notes and needs no additional information.

In order to simplify computational procedure the assessment coefficients given in Tables 4.8 and 4.9 are in the attached MATHCAD sheet "Mod_est.mcd" expressed using built-in distribution function of normal and Student t- distribution. In accordance with the principles of Annex D in [1] single variable and model representation of a resistance variable R are distinguished. The results shown in the attached sheet indicates that both approaches lead to similar results.

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Appendix 1 - Probabilistic models of basic variables

Distribution, notation	Probability density function	Domain of X	Parameters	Mean μ	Standard deviation σ	Skewness ω
Rectangular R(a,b)	$1/(b-a)$	$a \le x \le b$	a $b > a$	$(a+b)/2$	$(b-a)/\sqrt{12}$	0
Normal N(μ,σ)	$\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$	$-\infty \le x \le \infty$	μ σ	μ	σ	0
Lognormal, general LN(μ,σ,ω) LN(μ,σ,x_0)	$\frac{1}{\lvert x-x_0\rvert\sqrt{\ln(1+c^2)}\sqrt{2\pi}}\exp\left(-\left(\ln\frac{\lvert x-x_0\rvert\lvert c\rvert\sqrt{1+c^2}}{\sigma}\right)^2\Big/(2\ln(1+c^2))\right)$	$x_0 \le x < \infty$ pro $\omega > 0$, $-\infty < x \le x_0$ pro $\omega < 0$	$x_0 = \mu - c\sigma$ σ c	$x_0 + c\sigma$	σ	$3c+c^3$
Lognormal, zero origin LN(μ,σ)	$\frac{1}{x\sqrt{\ln(1+V^2)}\sqrt{2\pi}}\exp\left(-\left(\ln\frac{x\sqrt{1+V^2}}{\mu}\right)^2\Big/(2\ln(1+V^2))\right)$	$0 \le x < \infty$	μ $V = \sigma/\mu$	μ	$V\mu$	$3V+V^3$
Gamma Gam(μ,σ)	$\lambda^k x^{k-1}\exp(-\lambda x)/\Gamma(k)$	$0 \le x < \infty$	$\lambda = \mu/\sigma^2$ $k = (\mu/\sigma)^2$	k/λ	$\sqrt{k}/\lambda$	$2/\sqrt{k}$
Beta, general Beta(μ,σ,ω,b) Beta(μ,σ,a,b)	$\frac{(x-a)^{c-1}(b-x)^{d-1}}{B(c,d)(b-a)^{c+d-1}}$	$a \le x \le b$	a $b > a$ $c \ge 1$ $d \ge 1$	$a + \frac{(b-a)c}{c+d}$	$\frac{(b-a)}{cg+dg}$, $g=\sqrt{\frac{c+d+1}{cd}}$	$\frac{2g(d-c)}{c+d+2}$, $g=\sqrt{\frac{c+d+1}{cd}}$
Beta, zero origin Beta(μ,σ,ω) Beta(μ,σ,b)	$\frac{(x)^{c-1}(b-x)^{d-1}}{B(c,d)\,b^{c+d-1}}$	$0 \le x \le b$	$b > 0$ $c \ge 1$ $d \ge 1$	$\frac{b\ c}{c+d}$	$\frac{b}{cg+dg}$, $g=\sqrt{\frac{c+d+1}{cd}}$	$\frac{2g(d-c)}{c+d+2}$, $g=\sqrt{\frac{c+d+1}{cd}}$
Gumbel Gum(μ,σ)	$c\exp(-c(x-x_{mod}) - \exp(-c(x-x_{mod})))$	$-\infty \le x < \infty$	$x_{mod} = \mu - 0{,}577\sqrt{6}\sigma/\pi$ $c = \pi/(\sqrt{6}\sigma)$	$x_{mod} + 0{,}577/c$	$\pi/(\sqrt{6}c)$	1,14

Appendix 2 - Statistical parameters of functions of random variables

Function Z	The mean μ_Z	Standard deviation σ_Z	Skewness ω_Z
$aX+b$	$a\mu_X + b$	$\lvert a \rvert \sigma_X$	ω_X pro $\omega > 0$, - ω_X pro $\omega < 0$
X^2 *)	$\mu_X^2 + \sigma_X^2$	$2\sigma_X\left(\mu_X^2 + \mu_X \sigma_X \omega_X\right)^{1/2}$	$\dfrac{8\,\mu_X^3\,\sigma_X^3\left(\omega_X + 3\,V_X\right)}{\sigma_Z^3}$
$\dfrac{1}{X}$ *)	$\dfrac{1+V_X^2 - V_X^3\,\alpha_X}{\mu_X}$	$\dfrac{\left(V_X^2 - 2V_X^3\omega_X\right)^{1/2}}{\mu_X}$	$\dfrac{6\,V_X^4 - V_X^3\,\omega_X}{\mu_X^3\,\sigma_Z^3}$
$aX + bY + c$	$a\mu_X + b\mu_Y + c$	$\left(a^2\sigma_X^2 + b^2\sigma_Y^2\right)^{1/2}$	$\dfrac{a^3\,\sigma_X^3\,\omega_X + b^3\,\sigma_Y^3\,\omega_Y}{\sigma_Z^3}$
$X + Y$	$\mu_X + \mu_Y$	$\left(\sigma_X^2 + \sigma_Y^2\right)^{1/2}$	$\dfrac{\sigma_X^3\,\omega_X + \sigma_Y^3\,\omega_Y}{\sigma_Z^3}$
$X - Y$	$\mu_X - \mu_Y$	$\left(\sigma_X^2 + \sigma_Y^2\right)^{1/2}$	$\dfrac{\sigma_X^3\,\omega_X - \sigma_Y^3\,\omega_Y}{\sigma_Z^3}$
$X\,Y$ *)	$\mu_X\;\mu_Y$	$\mu_X\,\mu_Y\left(V_X^2 + V_Y^2 + V_X^2\,V_Y^2\right)^{1/2}$	$\dfrac{\mu_X^3\,\mu_Y^3\left(V_X^3\,\omega_X + V_Y^3\,\omega_Y + 6V_X^2\,V_Y^2\right)}{\sigma_Z^3}$
$\dfrac{X}{Y}$ *)	$\dfrac{\mu_X\left(1+V_Y^2 - V_Y^3\,\omega_Y\right)}{\mu_Y}$	$\dfrac{\mu_X\left(V_X^2 + V_Y^2 - 2\,V_Y^3\,\omega_Y\right)^{1/2}}{\mu_Y}$	$\dfrac{\mu_X^3\left(V_X^3\,\omega_X - V_Y^3\,\omega_Y + 6V_Y^4 + 6V_X^2\,V_Y^2\right)}{\mu_Y^3\,\sigma_Z^3}$

*) Expressions for parameters of marked functions are approximations only.

Appendix 3 - Fractile of a random variable x_p, $P(X \le x_p) = \Phi(x_p) = p$

Distribution, notation	Domain of X	Fractile x_p of the theoretical model $x_p =$	Estimate using coverage method		Estimate using prediction method	
			σ known	σ unknown	σ known	σ unknown
Rectangular R(a,b)	$a \le x \le b$	$a + p\,(b - a)$	-	-	-	-
Normal N(μ,σ)	$-\infty \le x \le \infty$	$\mu + u_p\,\sigma = \mu\,(1 + u_p\,V)$ u_p from Table 4.1	$m - \kappa_p\,\sigma$ κ_p from Table 4.3	$m - k_p\,s$ k_p from Table 4.4	$m+u_p(1/n+1)^{1/2}\,\sigma$ u_p from Table 4.1	$m+t_p(1/n+1)^{1/2}\,s$ $t_p(1/n+1)^{1/2}$ from Table 4.4
Lognormal, general LN(μ,σ,ω) LN(μ,σ,x_0)	$x_0 \le x < \infty$ pro $\omega > 0$, $-\infty < x \le x_0$ pro $\omega < 0$	$\mu - \frac{\sigma}{c}\left(1 - \frac{1}{\sqrt{1+c^2}}\exp\left(\mathrm{sign}(\alpha)u_p\sqrt{\ln(1+c^2)}\right)\right) =$ $= x_0 + \frac{\mu + x_0}{\sqrt{1+c^2}}\exp\left(\mathrm{sign}(\alpha)u_p\sqrt{\ln(1+c^2)}\right)$ u_p for normal distribution or $\mu + u_p\,\sigma = \mu\,(1 + u_p\,V)$ u_p for lognormal distribution from Table 4.2	$m - \kappa_p\,\sigma$ κ_p not given	$m - k_p\,s$ k_p from 4.5 and 4.6	$m+u_p(1/n+1)^{1/2}\,\sigma$ u_p from Table 4.2	$m+t_p(1/n+1)^{1/2}\,s$ t_p from Table 4.7
Lognormal, zero origin LN(μ,σ)	$0 \le x < \infty$	$\frac{\mu}{\sqrt{1+V^2}}\exp\left(u_p\sqrt{\ln(1+V^2)}\right) \cong$ $\cong \mu\,\exp\left(u_p \times V\right)$ for $V < 0{,}2$ u_p for normal distribution or $\mu + u_p\,\sigma = \mu\,(1 + u_p\,V)$ and u_p for lognormal distribution from Table 4.2	$m - \kappa_p\,\sigma$ κ_p not given	$m - k_p\,s$ k_p from Table 4.5 and 4.6	$m+u_p(1/n+1)^{1/2}\,\sigma$ u_p from Table 4.2	$m+t_p(1/n+1)^{1/2}\,s$ t_p from Table 4.7
Gumbel Gum(μ,σ)	$-\infty \le x < \infty$	$x_{\mathrm{mod}} - \frac{1}{c}\ln(-\ln(p)) \cong$ $\cong \mu - (0{,}45 + 0{,}78\ln(-\ln(p)))\,\sigma$	Fractile can be estimated using the parameter lognormal distribution as an approximation			

ATTACHEMENTS

1. MATHCAD sheet “DistFract.mcd” Fractiles of basic types of distributions.

MATHCAD sheet “DistFract.mcd” is intended for determination of fractiles of selected theoretical models.

2. MATHCAD sheet “SampFract.mcd” Estimation of sample fractile.

MATHCAD sheet “SampFract.mcd” is intended for determination of fractiles using limited samples.

3. MATHCAD sheet “Mod_est.mcd” Estimation of models.

MATHCAD sheet “Mod_est.mcd” is intended for determination of fractiles using limited samples taking into account model uncertainties.

Attachment 1 - MATHCAD sheet "DistFract.mcd"

MATHCAD sheet "DistFract" Fractiles of basic types of distributions

Definition of the fractile X_P : P = Prob ($X<X_P$), relative value $\xi_P = X_P/\mu_X$

1 Input data for a random variable X

Basic characteristics : $\mu := 1 \quad V := 0.10 \quad \sigma := V \cdot \mu$

An example of the design value for a resistance variable

$\alpha := 0.8 \quad \beta := 3.8 \quad P := \text{pnorm}(-\alpha \cdot \beta, 0, 1)$ check: $\boxed{P = 1.183 \times 10^{-3}}$

Range for the probability P considered below: $p := 0.001, 0.005 .. 0.999$

Standardised normal fractile given by inverse distribution function: $u(p) := \text{qnorm}(p, 0, 1)$

2 Fractiles of the normal distribution $\xi n(p) = X_P/\mu_X$

$$\xi n(p) := 1 + u(p) \cdot V$$

3 Fractiles of the two parameter lognormal distribution $\xi ln(p) = X_P/\mu_X$

$$\xi ln(p) := \frac{\exp\left(u(p) \cdot \sqrt{\ln\left(1 + V^2\right)}\right)}{\sqrt{1 + V^2}}$$ Correct formula for any V

$$\xi lna(p) := \exp(u(p) \cdot V)$$ Common approximation for $V < 0,2$

4 Fractiles of a general three parameter lognormal distribution

Skewness *a as a range variable* $a := -1, -0.5 .. 1$

Parameter C of three parameter lognormal distribution of *g*:

$$C(a) := \frac{\left(\sqrt{a^2 + 4} + a\right)^{\frac{1}{3}} - \left(\sqrt{a^2 + 4} - a\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}}$$

Parameters of transformed variable: $mg(a) := -\ln(|C(a)|) + \ln(\sigma) - (0.5) \cdot \ln\left(1 + C(a)^2\right)$

$sg(a) := \sqrt{\ln\left(1 + C(a)^2\right)} \qquad x0(a) := \mu - \frac{1}{C(a)}\sigma$ Check: $\boxed{x0(1) = 0.69}$

$$\xi lng(p, a) := 1 - \frac{V}{C(a)} \cdot \left(1 - \frac{\exp\left(\text{sign}(a)\, u(p) \cdot \sqrt{\ln\left(1 + C(a)^2\right)}\right)}{\sqrt{1 + C(a)^2}}\right)$$

5 Fractiles of the gamma distribution

Parameters of gamma distribution: $k := \left(\frac{\mu}{\sigma}\right)^2 \quad \lambda := \left(\frac{\mu}{\sigma^2}\right)$

Transformed variable u = λ x, shape factor s = k

No explicit formula is available $\xi gam(p) := \frac{qgamma(p, k)}{\lambda}$

6 Fractiles of the Gumbel distribution

Explicit formula: $\xi gum(p) := 1 - V \cdot (0.45 + 0.78 \ln(-\ln(p)))$

7 Relative values of fractiles $\xi_P = x_P/\mu_X$ versus probability P

Lower fractiles

The coefficient of variation: $\boxed{V = 0.1}$

Check:

$\boxed{\xi n(0.001) = 0.691}$

$\boxed{\xi ln(0.001) = 0.731}$

$\boxed{\xi lna(0.001) = 0.734}$

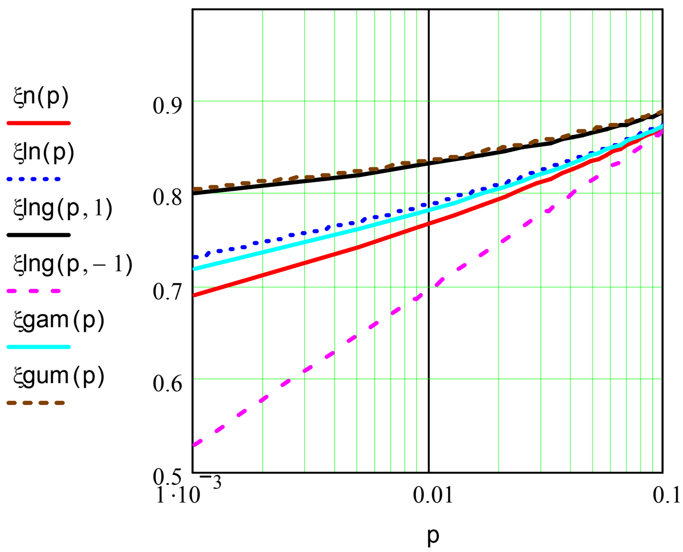
$\boxed{\xi lng(0.001, 1) = 0.801}$

$\boxed{\xi lng(0.001, -1) = 0.53}$

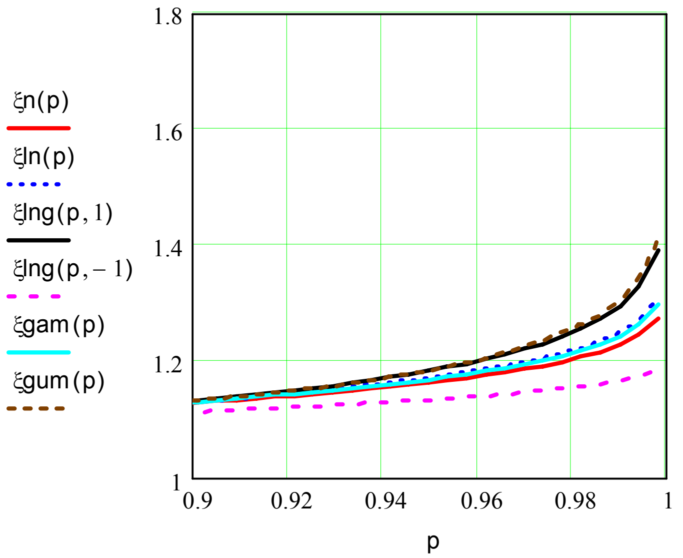
$\boxed{\xi lng(0.001, 0.4) = 0.743}$

$\boxed{\xi gam(0.001) = 0.719}$

$\boxed{\xi gum(0.001) = 0.804}$



Upper fractiles



Notes. 1) It follows from Figure that the skewness of the distribution may have significant effect on assessment of the design value (0,001 fractile).

2) Approximate formula for two parameter lognormal distribution yields sufficiently accurate results for the coefficient of variation V < 0,2.

3) Gamma and Gumbel distribution can be well approximated by three parameter lognormal distribution having skewness equal to $\alpha = 2*V$ and $\alpha = 1{,}14$ respectively.

Attachment 2 - MATHCAD sheet "SampFract.mcd"

MATHCAD sheet "SampFract" for estimation of sample fractile

MATHCAD sheet for determination of the characteristic and design values and material partial factor γM using test data in accordance to EN 1990, Annex D.

1. Analytic expressions for coefficients of fractile estimation given in EN 1990, Annex D

5% fractile V unknown $\quad ks(n) := qt(0.95, n-1)\sqrt{1+\frac{1}{n}}$

5% fractile V known $\quad k\sigma(n) := qnorm(0.95, 0, 1)\cdot\sqrt{1+\frac{1}{n}}$

0,1 % fractile V unknown $\quad ds(n) := qt(0.999, n-1)\sqrt{1+\frac{1}{n}}$

0,1 % fractile V known $\quad d\sigma(n) := qnorm(0.999, 0, 1)\sqrt{1+\frac{1}{n}}$

2. Characteristic and design values (relative values related to the mean)

Single variable, V unknown: xk=ξks*μx, xd=ξds*μx

$$\xi ks(n,V) := \frac{\exp\left[(-ks(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}} \qquad \xi ds(n,V) := \frac{\exp\left[(-ds(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}}$$

Single variable, V known: xk=ξkσ*μx, xd=ξdσ*μx

$$\xi k\sigma(n,V) := \frac{\exp\left[(-k\sigma(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}} \qquad \xi d\sigma(n,V) := \frac{\exp\left[(-d\sigma(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}}$$

Model, V unknown: xk=ξks*μx,xd=ξds*μx, weighting factors: $\alpha rt := 0.707 \quad \alpha\delta := 0.707$

$$\xi ksmod(n,V) := \exp\left[-\left(1.65\alpha rt^2 + ks(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right]$$

$$\xi dsmod(n,V) := \exp\left[-\left(3.09\alpha rt^2 + ds(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right]$$

Model, V unknown: xk=ξkσ*μx, xd=ξdσ*μx

$$\xi k\sigma mod(n,V) := \exp\left[-\left(1.65\alpha rt^2 + k\sigma(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right]$$

$$\xi d\sigma mod(n,V) := \exp\left[-\left(3.09\alpha rt^2 + d\sigma(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right]$$

3. Partial factor γM

Check values:

Single variable, V unknown $\quad \gamma Ms(n,V) := \frac{\xi ks(n,V)}{\xi ds(n,V)} \quad \boxed{\gamma Ms(100, 0.162) = 1.278}$

Single variable, V known $\quad \gamma M\sigma(n,V) := \frac{\xi k\sigma(n,V)}{\xi d\sigma(n,V)} \quad \boxed{\gamma M\sigma(100, 0.162) = 1.263}$

Model, V unknown $\quad \gamma Msmod(n,V) := \frac{\xi ksmod(n,V)}{\xi dsmod(n,V)} \quad \boxed{\gamma Msmod(100, 0.162) = 1.271}$

Model, V known $\quad \gamma M\sigma mod(n,V) := \frac{\xi k\sigma mod(n,V)}{\xi d\sigma mod(n,V)} \quad \boxed{\gamma M\sigma mod(100, 0.162) = 1.264}$

4. Graphs for characteristic and design values

V := 0.01, 0.02.. .4

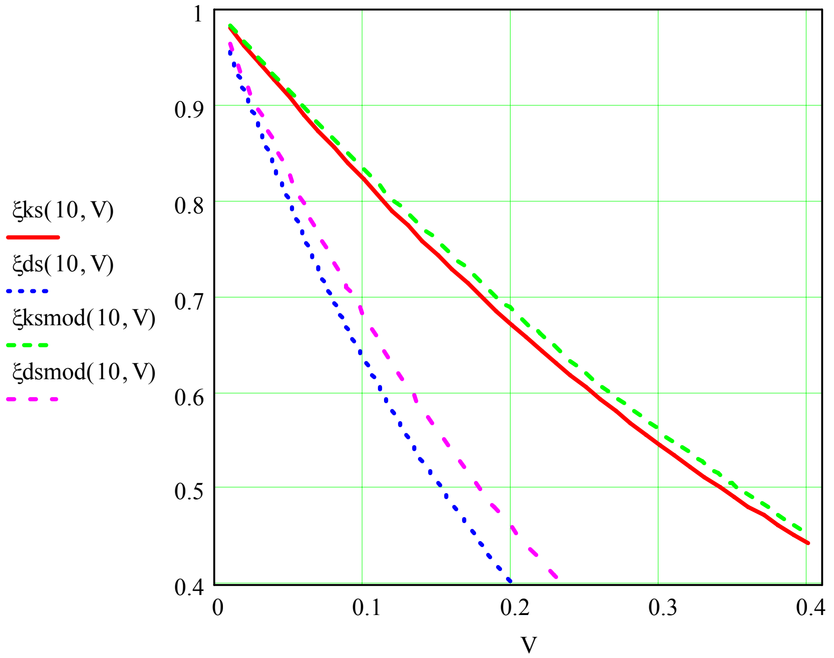


Figure 1. Characteristic and design values versus coefficient of variation V for n = 100 and weighting factors αrt= 0,301, $\alpha\delta$= 0,955

5. Graphs for γM factors

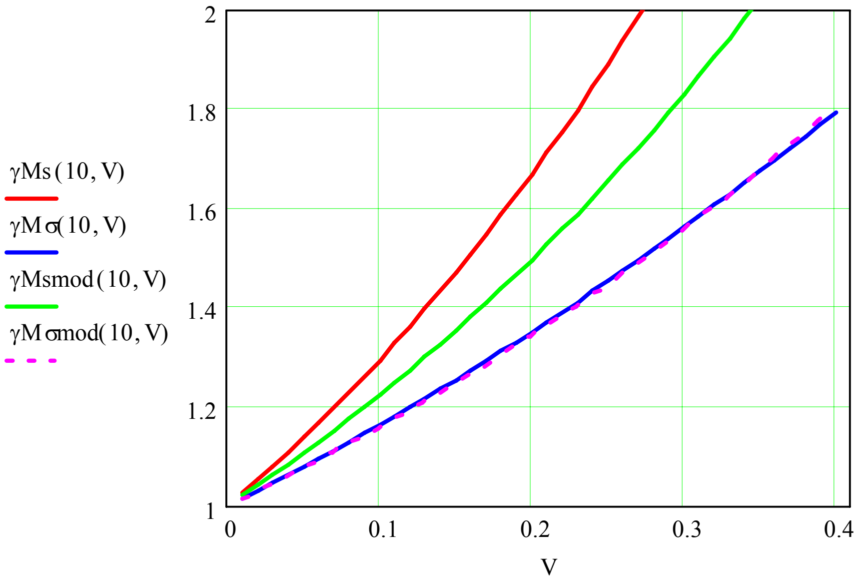


Figure 2. Partial factor γM versus coefficient of variation V for n=100 and weighting factors αrt= 0,301, $\alpha\delta$= 0,955

Attachment 3 - MATHCAD sheet “Mod_est.mcd”

MATHCAD sheet "Mod_est" Estimation of Models, characteristic, design and γM values determined from test data

MATHCAD sheet for estimation of models, the characteristic and design values of resistance variable R and material partial factor γM using test data (file "rdata.prn") in accordance to EN 1990, Annex D.

1. Experimental data

Test data to run the sheet without data recorded in the file "rdata.prn":

$rt := (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)^T$

$re := (1\ 3\ 4\ 4\ 5\ 5\ 6\ 8\ 10\ 9)^T$

Reading experimental data from the file "rdata.prn" located in the same directory

$DATA := READPRN("rdata.prn")$

Check values:

$rt := DATA^{\langle 0 \rangle}$ $\quad re := DATA^{\langle 1 \rangle}$ $\quad nr := length(rt)$ $\quad \boxed{nr = 21}$

The means of rt and re $\quad mre := mean(re)$ $\quad mrt := mean(rt)$ $\quad \boxed{mre = 0.719}$ $\boxed{mrt = 0.705}$

The least square fit for y=a+bx $\quad b := slope(rt, re)$ $\quad a := intercept(rt, re)$ $\quad \boxed{b = 0.975}$ $\boxed{a = 0.032}$

The least square fit for y=bx, a = 0 $\quad b := \frac{re \cdot rt}{rt \cdot rt}$ $\quad \boxed{b = 1.014}$

2. Check of experimental data taken from the file "rdata.prn" and check of the least square fit - slope b

The error terms δi

$\delta := \overrightarrow{\left(\frac{re}{b \cdot rt}\right)}$ $\quad \Delta := \overrightarrow{\ln(\delta)}$

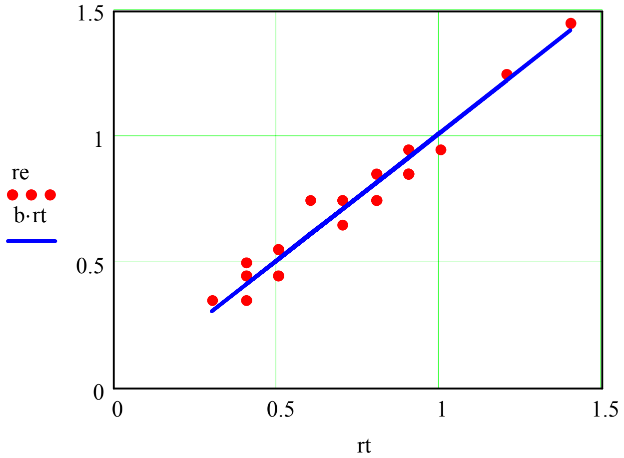


Figure 1. Experimental data and the line re = b rt.

Characteristics of Δ: $\quad m\Delta := mean(\Delta)$ $\quad s\Delta 2 := \frac{nr \cdot var(\Delta)}{nr - 1}$

Check values:

Coeff of variation of error terms Vδ $\quad V\delta := \sqrt{\exp(s\Delta 2) - 1}$ $\quad \boxed{V\delta = 0.107}$

Coefficient of variation of the model variables X1, X2, ...:

For example: $\quad VX1 := 0.08 \quad VX2 := 0.05$

Model coeff. of variation: $Vrt := \sqrt{VX1^2 + VX2^2}$ $\quad \boxed{Vrt = 0.094}$

The total coefficient of variation $\quad Vr := \sqrt{V\delta^2 + Vrt^2}$ $\quad \boxed{Vr = 0.143}$

Standard deviations

$Qrt := \sqrt{\ln(Vrt^2 + 1)}$ $\quad Q\delta := \sqrt{\ln(V\delta^2 + 1)}$ $\quad Q := \sqrt{\ln(Vr^2 + 1)}$ $\quad \boxed{Q = 0.142}$

Weighting factors: $\quad \alpha rt := \frac{Qrt}{Q}$ $\quad \alpha\delta := \frac{Q\delta}{Q}$ $\quad \boxed{\alpha rt = 0.663}$ $\boxed{\alpha\delta = 0.752}$

3. Coefficients of fractile estimation given in EN 1990, Annex D

$n := 3, 3.5 .. 30$

Check values:

5% fractile V unknown $\quad ks(n) := qt(0.95, n-1)\sqrt{1+\frac{1}{n}}$ $\quad \boxed{ks(10) = 1.923}$

5% fractile V known, appr. $\quad k\sigma(n) := qnorm(0.95, 0, 1)\cdot\sqrt{1+\frac{1}{n-2}}$ $\quad \boxed{k\sigma(10) = 1.745}$

0,1 % fractile V unknown $\quad ds(n) := qt(0.999, n-1)\sqrt{1+\frac{1}{n}}$ $\quad \boxed{ds(10) = 4.507}$

0,1 % fractile V known, appr. $\quad d\sigma(n) := qnorm(0.999, 0, 1)\sqrt{1+\frac{1}{n-2}}$ $\quad \boxed{d\sigma(10) = 3.278}$

4. Characteristic and design values (relative values related to the mean

$V := 0.0, 0.001 .. .4$

Note that the range variable V is generally used for the coefficient of variation of a single variable V and for a model investigation Vr, these may be different, for example V = 0,12 and Vr = 0,142.

Single variable, V unknown: rk=ξks*mr, rd=ξds*mr

$$\xi ks(n,V) := \frac{\exp\left[(-ks(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}} \qquad \xi ds(n,V) := \frac{\exp\left[(-ds(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}}$$

Single variable, V known: rk=ξkσ*mr, rd=ξdσ*mr

$$\xi k\sigma(n,V) := \frac{\exp\left[(-k\sigma(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}} \qquad \xi d\sigma(n,V) := \frac{\exp\left[(-d\sigma(n))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}}$$

Model, V unknown: rk=ξks*rx, xd=ξds*mr, weighting factors αrt and $\alpha\delta$ taken from the above experimental data re and rt (for V < 0,4 approximately Q =~ V):

Check values:

$$\xi ksmod(n,V) := \exp\left[-\left(1.65\alpha rt^2 + ks(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right] \qquad \boxed{\xi ksmod(nr, Vr) = 0.774}$$

$$\xi dsmod(n,V) := \exp\left[-\left(3.09\alpha rt^2 + ds(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right] \qquad \boxed{\xi dsmod(nr, Vr) = 0.608}$$

Model, V known: rk=ξkσ*mx, rd=ξdσ*mr, (for V < 0,4 approximately Q =~ V):

$$\xi k\sigma mod(n,V) := \exp\left[-\left(1.65\alpha rt^2 + k\sigma(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right] \qquad \boxed{\xi k\sigma mod(10, 0.2) = 0.696}$$

$$\xi d\sigma mod(n,V) := \exp\left[-\left(3.09\alpha rt^2 + d\sigma(n)\cdot\alpha\delta^2 + 0.5\cdot V\right)V\right] \qquad \boxed{\xi d\sigma mod(10, 0.2) = 0.516}$$

5. Estimates of the partial factors γM

Check values:

Single variable, V unknown $\quad \gamma Ms(n,V) := \frac{\xi ks(n,V)}{\xi ds(n,V)}$ $\quad \boxed{\gamma Ms(10, 0.1) = 1.294}$

Single variable, V known $\quad \gamma M\sigma(n,V) := \frac{\xi k\sigma(n,V)}{\xi d\sigma(n,V)}$ $\quad \boxed{\gamma M\sigma(10, 0.1) = 1.165}$

Model, V unknown $\quad \gamma Msmod(n,V) := \frac{\xi ksmod(n,V)}{\xi dsmod(n,V)}$ $\quad \boxed{\gamma Msmod(nr, Vr) = 1.273}$

Model, V known $\quad \gamma M\sigma mod(n,V) := \frac{\xi k\sigma mod(n,V)}{\xi d\sigma mod(n,V)}$ $\quad \boxed{\gamma M\sigma mod(nr, Vr) = 1.234}$

6. The relative characteristic and design values, the model values estimated using weighting factors determined above from experimental data given in the file "rdata.prn"

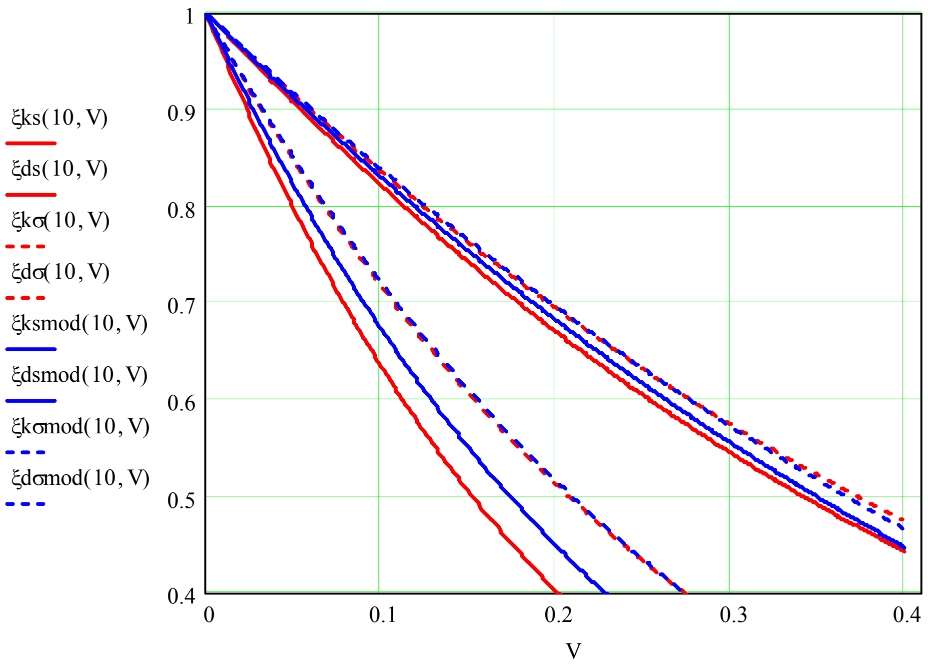


Figure 2. Variation of the characteristic and design values with coefficient of variation V for n = 10.

The char. value of X: $rks(n, V) := b \cdot mrt \cdot \xi ks(n, V)$ Example $\boxed{rks(21, 0.12) = 0.575}$

The model char. value of X: $rksmod(n, V) := b \cdot mrt \cdot \xi ksmod(n, V)$ From data: $\boxed{rksmod(nr, Vr) = 0.553}$

7. Partial factor γM, the model values estimated using weighting factors determined above from experimental data given in the file "rdata.prn"

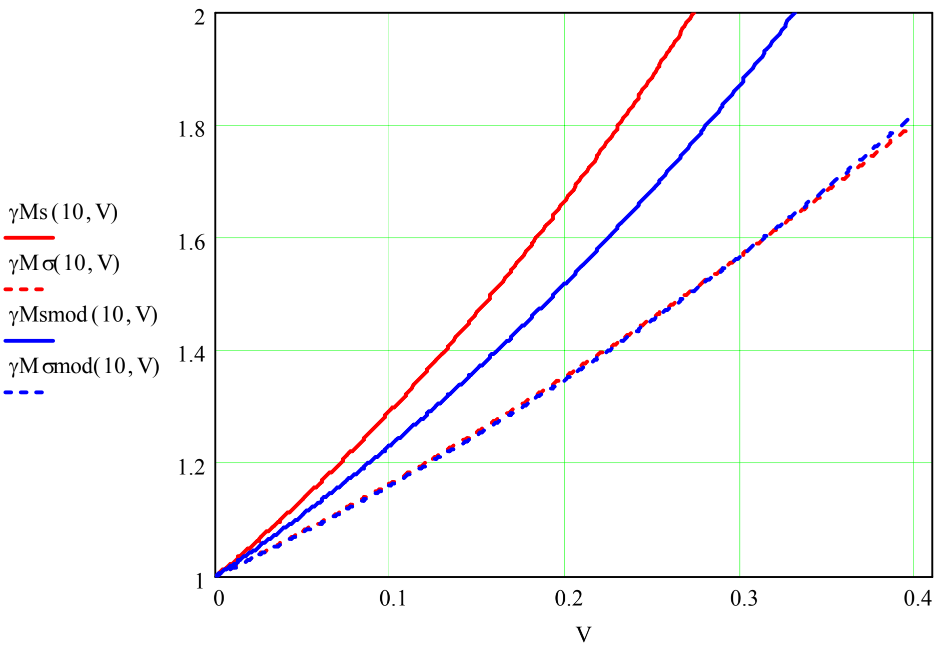
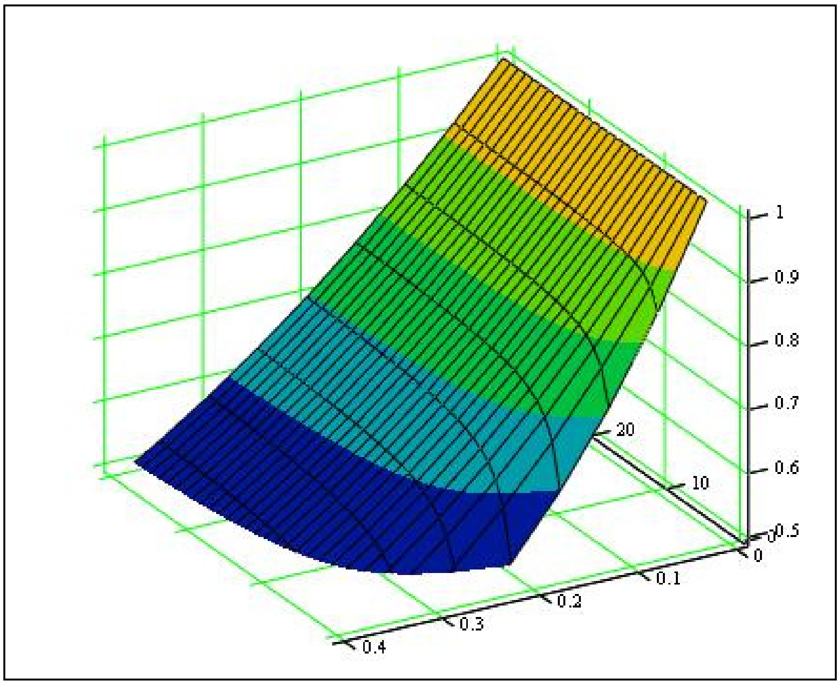


Figure 3. Variation of the partial factor γM versus coefficient of variation V for n=10.

8. Variation of ξks and γMs with n and V for the model values estimated using weighting factors determined above from experimental data given in the file "rdata.prn".

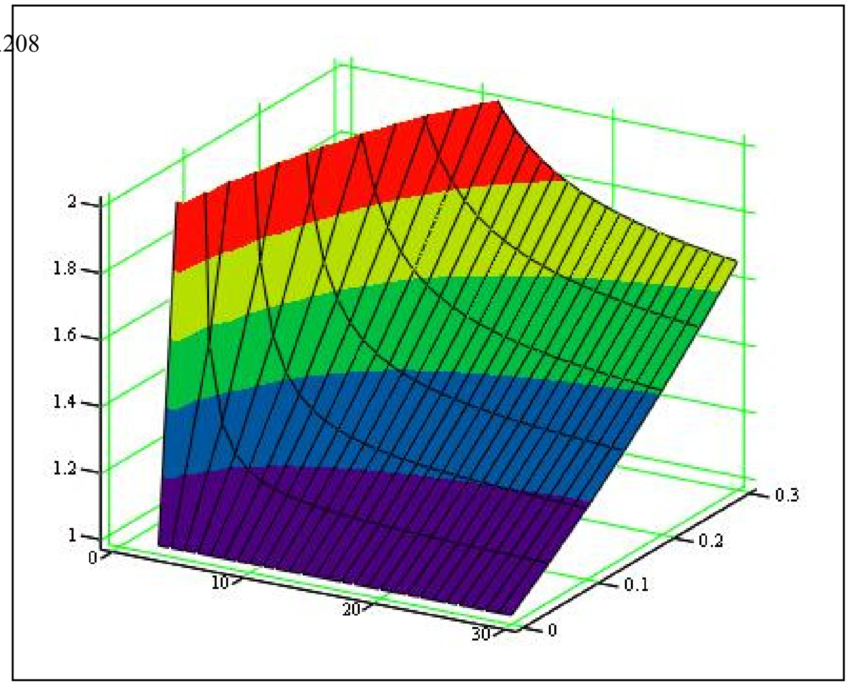
$\xi ks(3, 0.0) = 1$



ξks

Figure 4. Variation of the characteristic values ξks with n and V
.

$\gamma Ms(20, 0.1) = 1.208$



γMs

Figure 5. Variation of the partial factor γM with n and V .

ANNEX B – ELEMENTARY METHODS OF STRUCTURAL RELIABILITY II

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Summary

Using basic principles of the reliability theory described in Chapter II Elementary methods of structural reliability I, the operational techniques for estimating partial factors of basic variables are derived and applied to common permanent and variable loads. The described computational procedures are illustrated by a number of numerical examples, which are supplemented by MATHCAD, EXCEL and MATHEMATICA sheets.

1 INTRODUCTION

1.1 Background materials

Fundamental concepts and procedures of structural reliability are well described in a number of national standards, in the new European document EN 1990 [1] and International Standard ISO 2394 [2]. Additional information may be found in the background document developed by JCSS [3] and in recently published handbook to EN 1990 [4]. Guidance on application of the probabilistic methods of structural reliability may be found in publications and working materials developed by JCSS [5] and in relevant literature listed in [4] and [5]. Guidance on structural systems and time dependent reliability may be found in [6] and [7].

1.2 General principles

The theory of structural reliability considers all basic variables as random quantities having appropriate types of probability distribution. Different types of distributions should be considered for actions, material properties and geometrical data. In addition, model uncertainties of actions and resistance models should be taken into account. Prior theoretical models of basic variables and procedures for probabilistic analysis are indicated in JCSS documents [5].

This appendix is a direct extension of Chapter II "Elementary methods of structural reliability" of the main text to which reference is frequently made.

2 DESIGN POINT

Theoretical principles can be utilized in estimation of essential reliability elements (partial factors, reduction factors, combination rules, etc.) used in operational standards including Eurocodes that are based on a partial factor method. To explain how theoretical findings are transmitted into design recommendations a graphical representation of the random variables E and R and the corresponding limit state function (2) of Chapter II R - E = 0 is used similarly as in Annex C of EN 1990 [1].

Figure 1 (adopted from [1]) shows the random variables E and R in a two-dimensional diagram, where the horizontal axis indicates the fraction R/σ_R, the vertical axis the fraction E/σ_E. It is assumed that E and R are mutually independent variables having a normal distribution. As indicated in Examples 5 and 6 of Chapter II such an assumption might not be entirely realistic and should be considered as an approximation only. However, generally any distribution may be transformed to the normal distribution (at least in some domain) and, therefore, the random variables E and R in Figure 1 may be considered as transformed variables having originally other type of distribution.

Figure 1 also shows the limit state function (failure boundary) R - E = 0, which corresponds to expression (2) of Chapter II transformed to the coordinates used in Figure 1. Note that the failure boundary would be the diagonal of the main axes if the standard deviations of R and E have the same magnitude, $\sigma_R = \sigma_E$. The safe (desirable) domain of the variables R and E, where condition (1) of Chapter II is satisfied, is located under the failure boundary, the failure (undesirable) domain lies above the boundary $R - E = 0$.

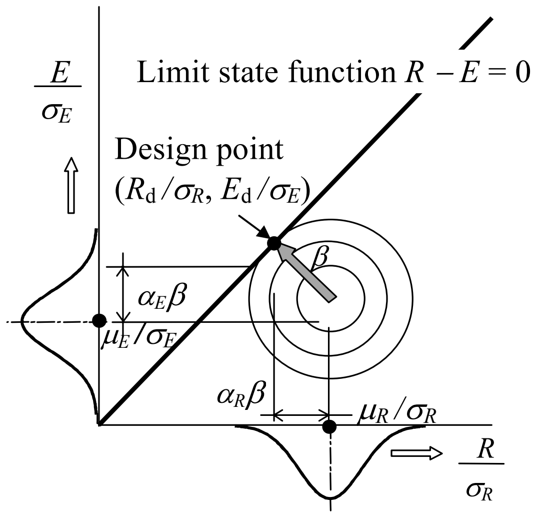


Figure 1. Design point R_d, E_d.

In verification of structural reliability any point on the failure boundary $R - E = 0$ could be considered as the critical (design) point (as apparent from the historical development of design methods described in previous chapter of this Handbook 2). However, it has been proved (see for example [5]) that the best option (assuring consistency and invariance of the solution for a different formulation of the limit state function and different basic variables) is the point (R_d, E_d) closest to the mean (μ_E, μ_R), which is indicated in Figure 1. Accepting this finding, it follows directly from Figure 1 that the design point coordinates (R_d, E_d) may be written in the form

$$R_d = \mu_R - \alpha_R\beta\sigma_R \quad (1)$$

$$E_d = \mu_E - \alpha_E\beta\sigma_E \quad (2)$$

Here α_E and α_R denote the so-called sensitivity factors of variables E and R. The “minus“ signs in equations (1) and (2) are conventionally accepted in the documents CEN [1] and ISO [2].

It follows from Figure 1 that the sensitivity factors α_E and α_R (direction cosines of the failure boundary) can be written as

$$\alpha_E = -\sigma_E / \sqrt{\sigma_E^2 + \sigma_R^2} \tag{3}$$

$$\alpha_R = \sigma_R / \sqrt{\sigma_E^2 + \sigma_R^2} \tag{4}$$

In Eurocodes an approximation of these sensitivity factors by fixed values is further accepted

$$\alpha_R = \sigma_R / \sqrt{\sigma_E^2 + \sigma_R^2} = 0{,}8 \tag{5}$$

$$\alpha_E = -\sigma_E / \sqrt{\sigma_E^2 + \sigma_R^2} = -0{,}7 \tag{6}$$

The validity of such an approximation is delimited by means of a condition for the ratio of the standard deviations in the form

$$0{,}16 < \sigma_E \,/\, \sigma_R < 7{,}6 \tag{7}$$

When this condition is not satisfied, then the sensitivity factor $\alpha = \pm 1{,}0$ is recommended to be used for the variable having a greater standard deviation. Let us remark that this simplification is on the safe side as the sum of squared direction cosines should be equal to one.

The design values E_d and R_d of the variables E and R are thus defined as fractiles of normal distribution

$$\mathrm{P}(E > E_\mathrm{d}) = \Phi_\mathrm{U}(+\alpha_E\beta) = \Phi_\mathrm{U}(-0{,}7\beta) \tag{8}$$

$$\mathrm{P}(R < R_\mathrm{d}) = \Phi_\mathrm{U}(-\alpha_R\beta) = \Phi_\mathrm{U}(-0{,}8\beta) \tag{9}$$

where $\Phi_U(u)$ denotes a standardized normal distribution. If $\beta = 3{,}8$, then the design values e_d and r_d are fractiles corresponding approximately to probabilities 0,999 and 0,001. Note that in equation (9) the use of the symmetry of normal distribution is taken into account, i.e. of the relationship $1 - \Phi_U(+\alpha_E\beta) = \Phi_u(-\alpha_E\beta)$.

When the load or resistance model contains several basic variables (other loads, more materials, geometrical data), equations (8) and (9) holds only for the leading variables (the most significant for the studied condition of reliability). For other (accompanying) variables the requirements on design values are reduced and it holds

$$\mathrm{P}(E > E_\mathrm{d}) = \Phi_\mathrm{U}(+0{,}4\alpha_E\beta) = \Phi_\mathrm{U}(-0{,}28\beta) \tag{10}$$

$$\mathrm{P}(R < R_\mathrm{d}) = \Phi_\mathrm{U}(-0{,}4\alpha_R\beta) = \Phi_\mathrm{U}(-0{,}32\beta) \tag{11}$$

When $\beta = 3{,}8$ the design values of accompanying (non-leading) variables are fractiles approximately corresponding to probabilities 0,9 and 0,1.

Design values are the upper fractiles (for actions) or the lower fractiles (for resistance), corresponding to certain probabilities of being exceeded (actions) or not reached (resistance). For leading variables, the probabilities are given by the distribution function of the normal standardized distribution for values $u = +\alpha_E\beta$ and $-\alpha_R\beta$, in the case of non-leading variables for reduced values $u = +0{,}4\alpha_E\beta$ and $-0{,}4\alpha_R\beta$. These probabilities (for the lower fractile approximately 0,001 for leading and 0,1 for accompanying variables) then serve to determine the design values even for those variables, which do not have normal distribution. Let us note that according to the general principles, it is necessary in the case of upper fractiles (actions) to consider the complementary probabilities (close to the value 1).

Example 1.

The design values E_d and R_d of variables E and R from Example 4 will be assessed assuming that the reliability index $\beta = 3{,}8$, $\alpha_E = -0{,}7$ and $\alpha_R = 0{,}8$. According to equation (8), it holds for E that

$$P(E > e_d) = \Phi_U(\alpha_E\beta) = \Phi_U(-2{,}66) = 0{,}0039$$

The complementary probability is therefore 0,9961 and we obtain from equation

$$e_d = \mu - (0{,}45 + 0{,}78\ln(-\ln(p)))\sigma = 50 - (0{,}45 + 0{,}78\times\ln(-\ln(0{,}9961)))\times 10 = 88{,}75$$

Note that when the normal distribution is assumed, it is given

$$e_d = \mu + u_p\sigma = 50 + 2{,}66 \times 10 = 76{,}6$$

According to equation (9), it holds for R

$$P(R < R_d) = \Phi_U(-\alpha_R\beta) = \Phi_U(-3{,}04) = 0{,}0012$$

For the lognormal distribution with the mean of 100 (units) and standard deviation of 10 (units) it follows

$$R_d \cong \mu \exp(u_{\mathrm{norm,p}} \times V) = 100 \times \exp(-3{,}04\times 0{,}10) = 73{,}79$$

For normal distribution we obtain

$$R_p = \mu + u_p\sigma = 50 - 3{,}04 \times 10 = 69{,}6$$

Obviously, it holds for the coordinates of the design point that $e_d > r_d$ and the tie rod does not satisfy the condition (1) of Chapter II (we know from the Example 4 of Chapter II that β is only 3,09). In order to satisfy the condition for a reliability index of 3,8, the parameters of variables E and R would have to be modified.

The attached MATHCAD sheets StRod.mcd, DesVRod.mcd may be used to make all numerical calculations.

3 PARTIAL FACTORS

3.1 Material properties

The above described reliability concepts may be used to assess the partial factors. The attached MATHCAD sheets GammaRGQ.mcd cover all computational procedures described below and may be used to make additional numerical calculations.

In accordance to EN 1990 [1] or ISO 2394 [2] the partial factor γ_R of material resistance R is defined as the fraction of its characteristic value R_k and the design value R_d, thus

$$\gamma_R = R_k / R_d \tag{12}$$

It is further assumed that the characteristic value R_k of a resistance variable R is defined as its 5% fractile [1], [2] and [5].

If a resistance variable R (strength) has a normal distribution, then the characteristic value R_k is given as

$$R_k = \mu_R - 1{,}645 \times \sigma_R = \mu_R(1 - 1{,}645 \times V_R) \tag{13}$$

The design value R_d of R can be estimated using the above derived equation (22) (see also documents [1] and [2]), thus

$$R_d = \mu_R - \alpha_R \times \beta \times \sigma_R = \mu_R - 0{,}8 \times \beta \times \sigma_R = \mu_R(1 - 0{,}8 \times \beta \times V_R) \tag{14}$$

In equation (13) and (14) μ_R denotes the mean, σ_R the standard deviation, V_R the coefficient of variation and $\alpha_R = 0{,}8$ the sensitivity factor of R.

Taking into account equations (13) and (14) it follows from (12) that the partial factor γ_R for a normal distribution of R can be assessed as

$$\gamma_R = (1 - 1{,}645 \times V_R) / (1 - 0{,}8 \times \beta \times V_R) \tag{15}$$

Assuming a lognormal distribution of R its characteristic value R_k can be determined [1], [2] using approximate equation

$$R_k = \mu_R \times \exp(- 1{,}645 \times V_R) \tag{16}$$

Similarly the design value R_d is approximated [1], [2] as

$$R_d = \mu_R \times \exp(-\alpha_R \times \beta \times V_R) \tag{17}$$

Taking into account equations (16) and (17) it follows from (12) that the partial factor γ_R for a lognormal distribution can be assessed as

$$\gamma_R = \exp(- 1{,}645 \times V_R) / \exp(-\alpha_R \times \beta \times V_R) \tag{18}$$

Figures 2 and 3 show the variation of the partial factor γ_R of the material property R with the reliability index β for selected values of the coefficient of variation V_R given for normal distribution by equation (15) (Figure 2) and lognormal distribution by equation (18) (Figure 3).

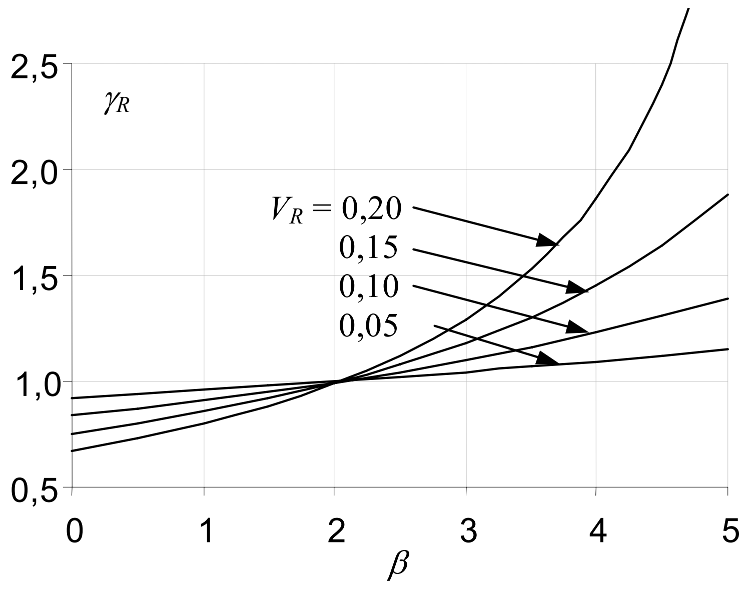


Figure 2. Variation of γ_R with β for selected coefficients of variation V_R = 0,05; 0,10; 0,15 and 0,20, and for normal distribution of R.

Generally the partial factor γ_R increases with increasing β. The increase of γ_R is considerably greater in the case of normal distribution (Figure 2) than in the case of lognormal distribution (Figure 3). The effect of the type of distribution is particularly obvious for coefficients of variation V_R greater than 0,10. A considerable effect of the type of distribution on the theoretical value of partial factors can be expected also for other basic variables, in particular for actions.

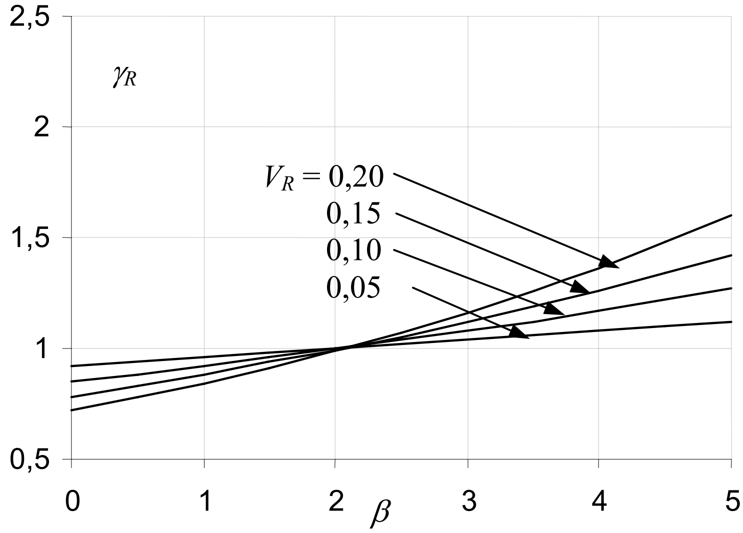


Figure 3. Variation of γ_R with β for selected coefficients of variation V_R = 0,05; 0,10; 0,15 and 0,20, and for lognormal distribution of R.

3.2 Permanent load

Consider a self-weight G having a normal distribution. Similarly as in the case of material property when a reference period T instead of the design working life T_d is used in the reliability verification of a structure, then the design value of G should be determined for T instead of T_d. It is assumed that the characteristic value G_k of G is defined as the mean μ_G [1], [2] and [5]:

$$G_k = \mu_G \tag{19}$$

The design value G_d is given by equation (23) of Chapter II (see also documents [1], [2]) as

$$G_d = \mu_G - \alpha_G \times \beta \times \sigma_G = \mu_G + 0{,}7 \times \beta \times \sigma_G = \mu_G(1 + 0{,}7 \times \beta \times V_G) \tag{20}$$

In equation (20) μ_G denotes the mean, σ_G the standard deviation, V_G the coefficient of variation and $\alpha_G = -0{,}7$ the sensitivity factor of G.

The partial factor γ_G of G is given as [1], [2]

$$\gamma_G = G_d / G_k \tag{21}$$

Taking into account equations (19) and (20) it follows from (21) that

$$\gamma_G = (1 + 0{,}7 \times \beta \times V_G) \tag{22}$$

Figure 4 shows the variation of the partial factor γ_G with the reliability index β for selected values of the coefficient of variation V_G = 0,05; 0,10; 0,15 and 0,20. Note that γ_G = 1,35 (recommended in EN 1990 [1]) corresponds approximately to the reliability index β = 3,8 if the coefficient of variation is about 0,1 (the value recommended in EN 1990 [1] was further increased by 5% to take into account model uncertainty).

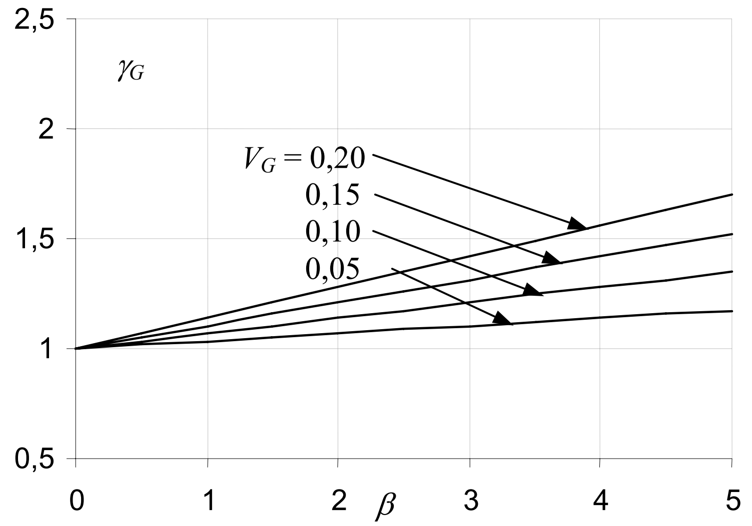


Figure 4. Variation of γ_G with β for selected coefficients of variation V_G = 0,05; 0,10; 0,15 and 0,20, and for normal distribution of G.

It follows from Figures 2, 3 and 4 that less significant variation with β -values should be generally expected for the partial factor of self-weight γ_G than for the partial factor of material property $\gamma_{R.}$.

3.3 Variable load

A similar procedure as in the case of permanent load G can be used for estimation of the partial factors γ_Q for variable loads Q. Assuming Gumbel distribution the characteristic value (0,98 fractile) is given as

$$Q_\mathrm{k} = \mu_Q\,(1 - V_Q\,(0{,}45 + 0{,}78\,\ln(-\ln(0{,}98)))) \tag{23}$$

The design value Q_d is given as

$$Q_\mathrm{d} = \mu_Q\,(1 - V_Q\,(0{,}45 + 0{,}78\,\ln(-\ln(\Phi^{-1}(-\alpha_E\beta)))) \tag{24}$$

In equation (23) and (24) μ_G denotes the mean, V_Q the coefficient of variation of annual extremes of Q and $\alpha_G = -\,0{,}7$ the sensitivity factor of Q.

The partial factor γ_Q of Q is given as [1], [2]

$$\gamma_Q = Q_\mathrm{d}\,/\,Q_\mathrm{k} \tag{25}$$

Figure 5 shows the variation of γ_Q with the coefficients of variation V_Q for selected values of β assuming Gumbel distribution of Q. It appears that in case of a variable action Q the reliability index β has a significant effect on the partial factor γ_Q. The following Figure 6 shows the variation of γ_Q with the reliability index β for selected values of the coefficients of variation V_Q assuming again Gumbel distribution of Q.

It follows from Figures 5 and 6 that for the reliability index β = 3,8 and the coefficient of variation V_Q up to 0,5, the partial factor γ_Q is less than 1,3. However the coefficient of variation may be also greater than 0,5 and other distribution may be more adequate (see other Chapters of this Handbook). That is why a conservative value γ_Q = 1,5 is recommended in EN 1990 [1].

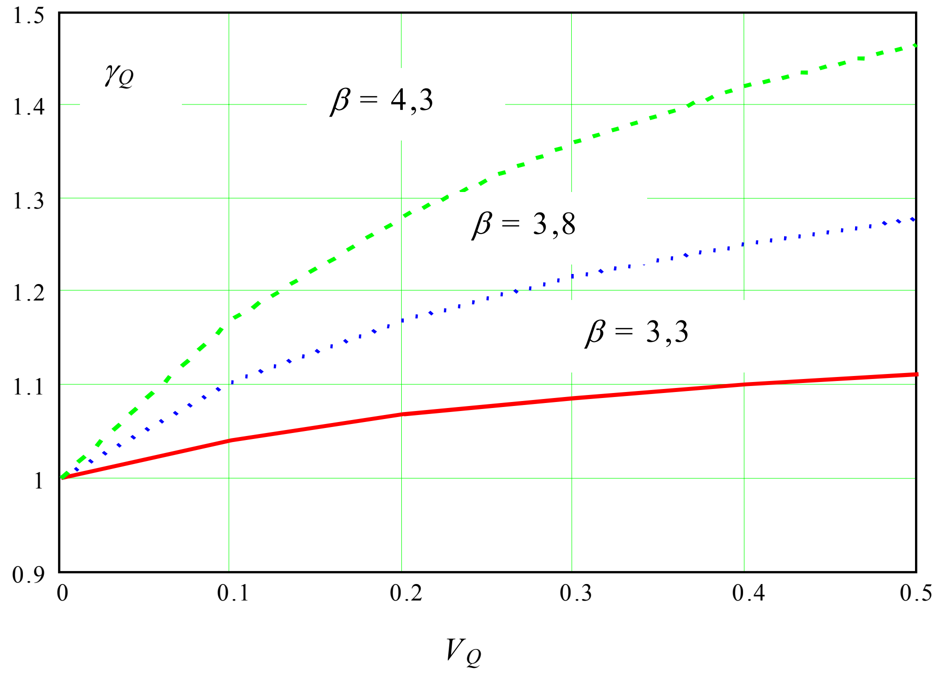


Figure 5. Variation of γ_Q with the coefficients of variation V_Q for selected values of β assuming Gumbel distribution of Q.

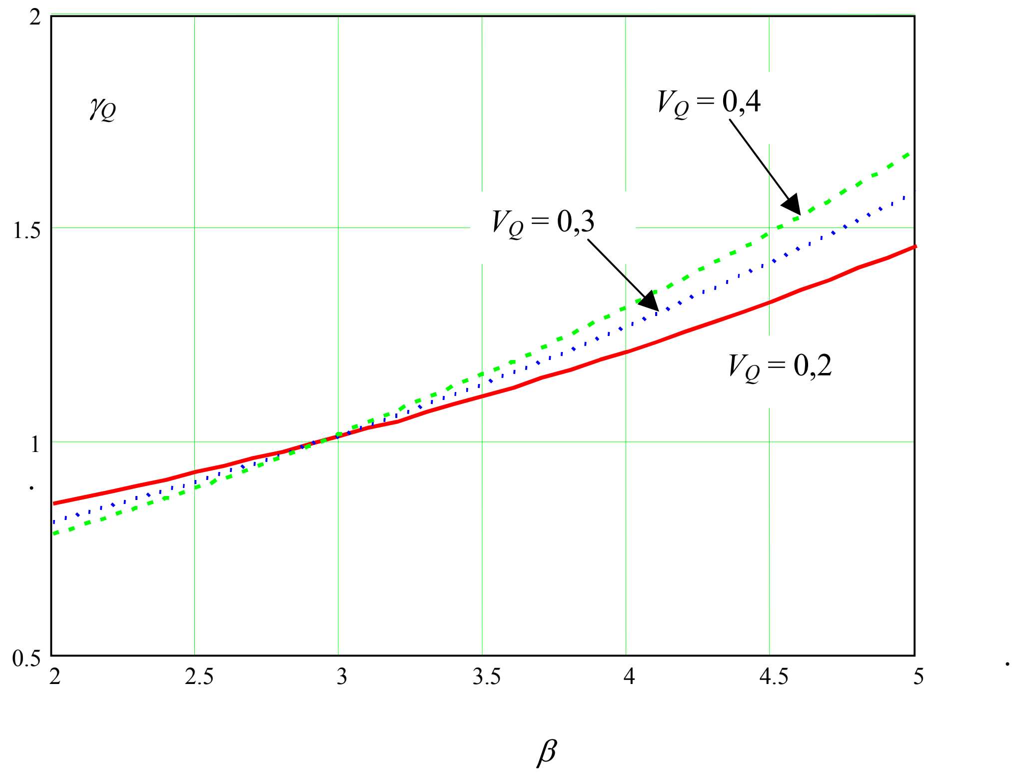


Figure 6. Variation of γ_Q with the reliability index β for selected values of the coefficients of variation V_Q assuming Gumbel distribution of Q.

It follows from Figure 6 that for the reliability index $\beta \cong 3$ the theoretical value of the partial factor γ_Q is about 1. This is due to the fact that the characteristic value Q_k is defined as 0,98 fractile, and when $\beta \cong 3$ then Q_k, is approximately equal to Q_d (for more details see equations (23) and (24)). Additional numerical calculations can be easily obtained using the attached MATHCAD sheets GammaRGQ.mcd, which include all the above described computational procedures.

4 THE GENERAL CASE OF RELIABILITY ANALYSIS

4.1 General

In the Chapter II "Elementary methods of structural reliability I" of this handbook it has been presented what is called the fundamental case of structural reliability. It is the case in which the limit state function can be represented by only two random independent variables, the effect of the actions and the resistance. This fundamental case represents a very interesting case for introducing the reliability concepts due to its very intuitive reasoning, and that we are used to this concepts of global action effect and resistance. Also, due to the fact that in two dimensions only is easy render graphic simple representations.

But, unfortunately, only in a few cases the limit state function may be represented by the fundamental case, reducing the structural reliability problem to a simple resistance versus action effect formulation assuming their independents. In most cases, at least, a few more variables will be needed. In general the resistance is a function of the material(s) properties and the dimensions of the structure or element and the action effects depend on the various applied loads and densities and dimensions of the structure. Even, not always the resistance and action effects can be considered independent: for instance, some dimensions affect both the actions and resistances. Even more, the action effects may depend on the response of the structure as a whole (e.g., the dynamic actions).

The limit state function can be written as:

$$Z(X_1, X_2, X_3, ...) = 0\ ; \tag{26}$$

or in vector form:

$$Z(\boldsymbol{X}) = 0\ ; \tag{27}$$

where $\boldsymbol{X} = \{X_1, X_2, X_3, ...\}$ is the vector of the random variables depending on time defining the limit state function.

In this case also, $Z(\boldsymbol{X}) > 0$ represents the safe region and $Z(\boldsymbol{X}) < 0$ the unsafe region. The probability of failure, in this case is obtained as:

$$P_f = \mathrm{P}\,[Z(\mathbf{X}) \leq 0] = \int\limits_{Z(X)<0}\!\!\ldots\!\int f_{\mathbf{Z}}(\mathbf{x})\, d\boldsymbol{x}; \tag{28}$$

where $f_{\mathbf{Z}}(\boldsymbol{x})$ is the joint density probability distribution of the vector of variables $\boldsymbol{X}$.

In the case of all the basic variables are independent, in a lot of cases we can adopt this hypothesis as a good approximation, the joint density probability distribution of $\boldsymbol{X}$ is the product of the marginal density probability distribution of each variable. Therefore, is possible to put the equation (18) in the form:

$$P_f = P\,[Z(\boldsymbol{X}) < 0] = \int\limits_{Z(X)<0}\!\!\ldots\!\int f_{X1}(x_1)\, f_{X2}(x_2)\, ...f_{Xn}(x_n)\ \mathrm{d}x_1\mathrm{d}x_2...\mathrm{d}x_n. \tag{29}$$

4.2 Basic variables

Basic variables are those variables needed to characterize and define the behaviour of a structure, and its safety, relating to a determined limit state.

The designer has some degree of freedom in order to choose the basic variables: In general, those used in the conventional design are chosen: dimensions, weights, loads, material strengths, etc. In general, independent basic variables are considered as any dependence adds complexity and it is difficult to define the degree of dependence. However, some variables as the tension, compression strength and the modulus of elasticity of the material are known to be dependent, but one can, generally, deal with them as they were independent.

In order to analyse structural reliability it is necessary to characterize basic variables statistically. That is, to define, at least, their distribution functions, parameters and correlation matrix. The parameters of distributions can be estimated on the basis of data by means of the usual statistical techniques: maximum likelihood, methods of moments, etc. The data should be carefully investigated in order to exclude the outliers, to analyse trends, and so on. A graphical representation of data and of the model adopted is generally useful.

Guidance concerning distribution functions and their parameters for the models of common actions and resistances in structural reliability are given in reference [5].

Example 1:

Consider a basic variable and the following experimental data:

data={1.3,3.2,4.3,1.3,5.4,3.7,3.8,4.0,2.9,3.2,4.5,4.0,3.4,2.4,1.8,1.7,2.2,4.1,2.6,4.1,3.3, 3.5,3.7,2.4,2.8,2.5,3.,3.3,2.6,2.9,2.4,2.6,2.9,2.8,3.1,3.2,3.4,3.5,3.2,3.3,3.};

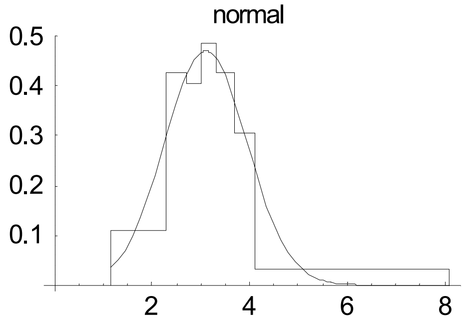
In the following the results obtained with the attached MATHEMATICA package distribution-fit are used. The following statistical characteristics were obtained:

Number of data	Mean	Variance	Standard variation	Coefficient of variation
41	3,105	0,700	0,837	0,270

We try to fit the normal, lognormal and gamma distributions to data. The parameters of the distributions are obtained by the method of moments. Assuming the following intervals: {1.3-2.3, 2.3-2.7, 2.7-3.3, -3.7, 3.7-4.1, 4.1-6}, the Chi square test is performed.

In the figure are represented the histogram and the density distribution function (in this case the normal distribution). The results obtained for each distribution are shown in the following table:

Distribution	Parameters	Estimator W	Confidence Level %
Normal	3,103- 0,847	0,612	96,0 - 99,6
Gamma	13,40 - 0,231	1,001	91,0 – 98,6
Lognormal	1,096 – 0,268	1,310	86,0 – 97,1



The range of the confidence level is due to the different hypothesis. The moments are known a priori, then the degrees of freedom are the number of intervals minus one; or obtained from data, then the degrees of freedom are the number of intervals minus three. Therefore to the same estimator correspond two confidence levels.

It appears that all three distributions fit well the data (normal distribution seems to be the best). Therefore, it is not clear what distribution to choose; plotting the cumulative distribution functions and the point values in direct representation and with the ordinates in double logarithmic scale, then it is possible to have a new insight in the problem.

4.3 Tail sensitivity problem

In statistics, the decisions are, generally, not taken after a "mathematical proof". A hypothesis is accepted if evidence for its rejection is not found. The probability distribution assigned to any variable can have an important influence on the estimated probabilities of failure. Besides, when assigning distribution functions, the data available are mainly, of course, of the central part of the distribution where the election of one or other distribution function is no very significant. This problem is called "tail sensitivity problem".

This problem is show graphically in Figure 7 where three distribution functions with the same mean and standard deviation, corresponding to the case of the Example 1: normal, gamma and log-normal, are represented both at normal scale and with double logarithmic ordinates. (i.e. $z = -\text{Log}(-\text{Log}(y))$), that is in Gumbel's paper, that increases the scale of the upper tail). At normal scale there are no significant difference among the distributions, but looking at the Gumbel's paper, the differences are little for the 0,95 fractile (i.e. characteristic value), while the differences are appreciable for the 0,999 fractile (i.e. design value). Also, on the Gumbel's paper may be seen that all distributions fit adequately to the points distribution, assuming for the ordered points a probability of i/n +1, but in order to estimate the design value (to say, fractile 0,999) it is necessary to extrapolate, and thus the difference among distributions is important. The higher point of the data corresponds to approximately to 0,975 fractile (1-1/42).

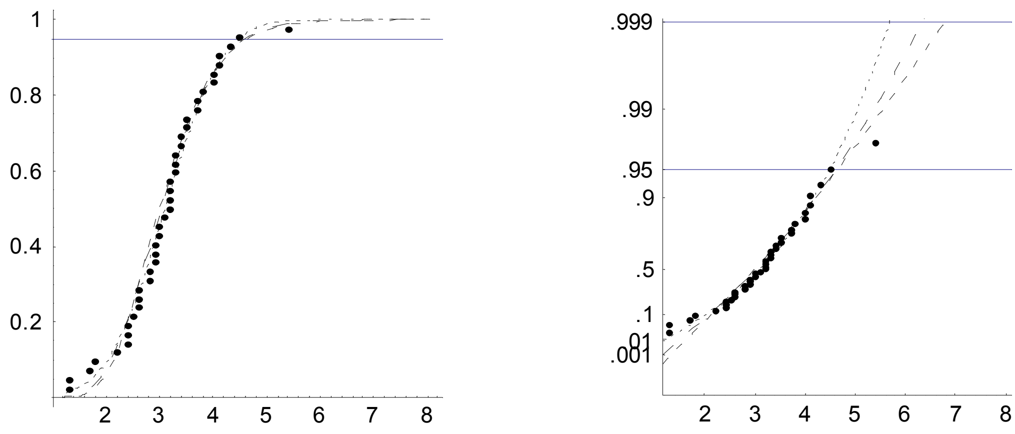


Figure 7. Cumulative distribution functions at normal scale and in Gumbel's paper.

5 AN EXAMPLE OF REINFORCED CONCRETE SLAB

5.1 General

Various design concepts mentioned above may be illustrated considering a simple example of a reinforced concrete slab in an office building. The example shows how different design methods (permissible stresses, global safety factor, partial factor method) treat uncertainties of basic variables by choosing different input (design) values. The example also indicates significance of the reliability theory in structural design and advantages of the reliability based partial factor method compared to the other design formats.

Some basic terms (e.g. characteristic strength) and calculation procedures used in this Section will be properly defined in various Chapters of this Handbook 2. Nevertheless the following text can be understood at least intuitively.

5.2 A reinforced concrete slab

A simply supported slab having the span of 6 m is exposed to a permanent load (self-weight of the slab and other fixed parts of the building), which is estimated by the characteristic value (equal to the mean value) g_k=7 kN/m^2. In accordance with the EN 1991-1-1 [8] the characteristic value of the imposed load in an office area q_k=3 kN/m^2 may be assumed. It is, however, well known that the mean value of this load is considerably lower, about 0,8 kN/m^2.

Further, the concrete C20/25 having the characteristic strength f_{ck}=20 MPa (the mean 30 MPa) and reinforcement bars S 500 having the characteristic strength f_{yk}=500 MPa (the mean 560 MPa) are to be used. Using previous experience, the total height of the slab 0,25 m (effective depth about 0,25 – 0,03 = 0,22 m) was specified in advance. Given the above data verification of the preliminary specifications and estimation of the necessary reinforcement area of the slab should be done.

5.3 Design of a slab

Consider first a simple drawing of the slab cross-section including simplified stress distribution diagrams in the compressive concrete zone (rectangular and triangular) as indicated in Figure 8.

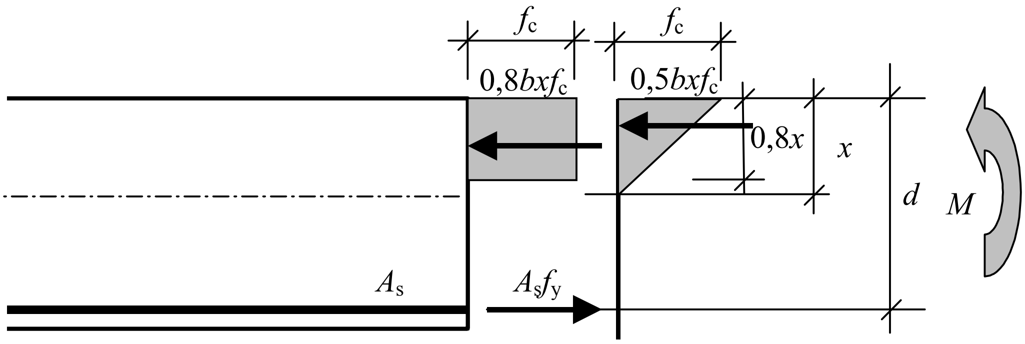


Figure 8. Stress distribution in a reinforced concrete slab.

When the rectangular stress diagram is assumed the following equilibrium conditions can be written (see Figure 8):

$$0{,}8\, f_c\, x\, b = A_s\, \mathrm{f_y} \tag{30}$$

$$A_s f_y (d - 0{,}4\, x) = M \tag{31}$$

The basic variables used in equations (30) and (31) are evident from Figure 8: d denotes the effective depth, x the depth of the neutral axis, b the width of the slab (considered as 1 m), A_s the area of the reinforcement, f_c the concrete strength and f_y the reinforcement strength (yield point). The bending moment in equations (31) is given as

$$M = \frac{(g+q)L^2}{8} \tag{32}$$

L denotes the span of the simply supported beam.

Using equilibrium conditions (29) and (30) the following formula for the reinforcement area A_s can be derived:

$$\frac{A_s}{bd} = \frac{f_c}{f_y}\left(1 - \sqrt{1 - \frac{2M}{f_c bd^2}}\right) \tag{33}$$

Without going into technical details note, that equation (33) may be used approximately for the global safety factor methods and partial safety factor method. Attached EXCEL sheet RCBeam and MATHCAD sheet RCBeam may be applied to make necessary calculations.

The classical permissible stresses method assumes a triangular compressive stress block in the compressive concrete zone, which is also indicated in Figure 8, and linear stress strain diagram. The reinforcement area A_s can be found using the attached MATHCAD sheet RCBeam.mcd, which includes further calculation details.

Note that up to now all the basic variables have been considered as deterministic quantities without considering any uncertainties that may potentially affect their actual values. However, it is well recognised that some of the basic variables entering equation (31) might have considerable scatter (particularly the load effect (bending moment) M, concrete strength f_c, and reinforcement strength f_y). On the other hand the geometric data A_s, b and d seem to be much less uncertain (almost fixed or deterministic).

To get a first estimate of the area $A_{s,}$ one may take the mean (average) values of all the basic variables involved. Intuitively it is clear that this might be not safe enough and instead of the mean values somehow “conservative (safe) values” should be applied. Table 1 indicates the mean values together with values of basic variables as used in design in accordance with the rules of above-mentioned design methods.

Table 1. Input design (characteristic) values of loads and material strengths used in design calculation using different design methods.

Basic variable	The mean value	Design method		
		Permissible stresses	Global safety factor	Partial factor
Permanent load g [kN/m^2]	7	7	7	9,45
Imposed load q [kN/m^2]	0,8	3	3	4,5
Concrete compressive strength f_c [MPa]	30	5,5	20	13,3
Reinforcement tensile strength f_y [MPa]	560	275	500	435

Table 1 clearly indicates the differences between considered design methods. For example, the input values of the permanent load g used in design calculation in accordance with the permissible stresses method and the global safety factor method are equal to the mean value (7 kN/m^2), while the design value in the partial factor method (9,45 kN/m^2) is obtained as a product of the characteristic value and the partial factor γ_G = 1,35. The design value of imposed load q used in design calculation in accordance with the permissible stresses method and the global safety factor method are equal to the characteristic value (3 kN/m^2), while the design value (4,5 kN/m^2) in the partial factor method is product of the characteristic value and partial factor γ_Q = 1,5.

The input calculation values for the compressive concrete strength f_c used by different design methods are considerably different from its mean (30 MPa). The same is true for the tensile strength of reinforcement f_y. Permissible stresses indicated in Table 1 can be found in design standards. Obviously they are much lower than the mean value as they are assumed to take into account all uncertainties including loads. Input strengths in the global factor methods are equal to the characteristic values. In the partial factor method the design strength of concrete 13,3 MPa is obtained by dividing the characteristic value 20 MPa by the partial factor γ_M = 1,5, similarly the design strength of steel 435 MPa is obtained as a fraction of the characteristic value 500 MPa and the partial factor γ_M = 1,15. Note that the factor 1,9 is used to enhance the load effect (bending moment M) when the global factor method is used to specify the reinforcement area A_s.

Resulting reinforcement areas obtained for all the design methods mentioned above are indicated in Table 2.

Table 2. A simply supported reinforced concrete slab designed using different historical methods assuming the span L = 6 m, the height h = 0,25 m (d = 0,22 m) and the loads g_k= 7 kN/m^2, q_k= 3 kN/m^2 (the mean 0,8 kN/m^2), C20/25 (f_{ck} = 20MPa, the mean 30 MPa), f_{yk}=500 MPa (the mean 560 MPa).

Design method	M [kNm]	A_s [m^2]	μM_R [kNm]	β	p_f
The mean value (absurd)	35,1	0,00038	35,1	0	0,5
The permissible stresses	45,0	0,00204	228,9	8.0	$0{,}44\times10^{-16}$
The global safety factor (s_0 =1,9)	45,0	0,00082	97,4	5,0	$0{,}32\times10^{-7}$
The partial factors method (CEN)	62,8	0,00069	82,4	4,2	$0{,}12\times10^{-5}$

The attached EXCEL sheet RCBeam and MATHCAD sheet RCBeam may be used to check the resulting data indicated in Table 2. These sheets also indicate further details of applied computational procedures.

It follows from Table 2 that resulting reinforcement areas vary within a broad range from A_s = 0,00038 m^2 (the mean value estimate) up to A_s = 0,00204 m^2 (the permissible stress method). The most economic procedure appears to be provided by the partial factor methods that leads to the lowest reinforcement area A_s = 0,00069 m^2.

5.4 Reliability consideration

The rectangular stress distribution (indicated in Figure 8), assumed in design calculation according to the global or the partial factor method, is accepted for reliability investigation of the slab designed by any of the design methods illustrated in Table 2 (including the permissible stress method, which assumes a triangular stress diagram). Thus, taking into account equilibrium conditions (30) and (31) the limit state function (16) may be written as

$$Z(\boldsymbol{X}) = A_s f_y \left(d - \frac{A_s f_y}{2bf_c}\right) - M = A_s f_y \left(d - \frac{A_s f_y}{2bf_c}\right) - \frac{(g+q)L^2}{8} \tag{34}$$

Assuming the limit state function (34) the reliability indexes β and the failure probabilities p_f can be determined using commercially available software VaP [9] or COMREL [10] or the MATHEMATICA notebook FORM.nb. An approximate value of the reliability index β and the failure probability p_f (with an acceptable accuracy) may be obtained using the attached MATHCAD sheet RelRCB, which is based on a simple procedure of numerical integration. The self-content MATHCAD sheet RelRCB is provided with explanatory notes (including information on input theoretical models of basic variables) and might be intuitively understood without any additional information. Detailed description of the applied numerical procedure is given in the other chapters of this Handbook 2.

Resulting reliability indexes β and the failure probabilities p_f are indicated in Table 2. The partial factor method leads to the reliability level described by β = 4,2 (failure probability 1,2 10^{-5}) that is very close to the value β = 3,8 (failure probability 7,2 10^{-5}) recommended in EN 1990 [1]. Slightly more conservative design (β = 5) is provided by the global safety factor methods (see Table 2). However, the permissible stresses method seems to lead to a rather uneconomical design (β = 8). Obviously, "the mean value method" proves to be unacceptable as it yields the lowest reinforcement area A_s = 0,00038 m^2 (reinforcement ratio 0,0022 only) corresponding to β = 0 and high probability of failure of P_f = 0,5.

6 ASSESEMENT OF THE FAILURE PROBABILITY IN GENERAL CASE

6.1 General

There are several different procedures for assessing failure probability in a general case of more variables:

- *Analytical*: Only in a few, very simple cases it is possible to find the analytical correct solution. Depends on the variables vector, all must be independent and normally distributed, and on the limit state region, it must be defined by hyper-planes. It cannot be considered as a general solution.
- *Numerical*: It is an exact solution in the sense that we can get, in principle, all the precision we need. The simple trapezoidal rule of integration gives, in general good results if there are not too many variables (4 or 5). The complexity of integration increases exponentially with the number of variables.
- *Monte Carlo methods*: The Monte Carlo simulation techniques are bases on the random sampling of the variables and carry a long number of artificial experiments. The use of this approach is increasing notably nowadays, due to bigger power and speed of the computers. The crude application of the method lead to the same difficulties explained above. Variance reduction and importance methods are employed to avoid those difficulties.
- *First and Second Order Reliability Methods (FORM and SORM)*: These approximate methods give iterative algorithms that allow to obtain the reliability index, using a linear or quadratic approximation to the limit state surface in the point of minimum distance to the mean point of the variables.

6.2 First and Second Order Reliability Methods (FORM and SORM)

Hassofer and Lind developed an algorithm invariant for the formulation of the limit state function. To obtain the reliability index the following steps have to be followed:

1. Define the limit state function.
2. Characterize statistically the basic variables; i.e.: mean, standard deviation, distribution function and correlation matrix.
3. Transform the set of basic variables into a set of independent variables (by means of the Rosenblatt's transformation, for instance).
4. Standardize the set of basic variables by the transformation $\boldsymbol{X}\rightarrow\boldsymbol{U}$, such that

$$\mathrm{E}\,(\boldsymbol{U}) = \mathbf{0}\text{ , and }\mathrm{CoV}[\boldsymbol{U},\boldsymbol{U}^{\mathrm{T}}] = 1\ . \tag{35}$$

5 Obtain the length of the minimum distance vector from the new origin point to the tangent hyper-plane of the limit state surface, referred to the new variables, in the intersection point of that vector with the limit state surface.

6 Obtain the design point, (X_{1d}, X_{2d},...,X_{nd}) and the sensitivity coefficients, α_1, α_2,...,α_n, the unitary cosines of that vector.

Figure 1 represents the space of the fundamental case, with only two variables: the action effect and the resistance. The origin of the β vector corresponds to the mean point of the variables $\boldsymbol{X}$, i.e.: (μ_{X1}, μ_{X2},...μ_{Xn}). The minimum distance vector from this point to the limit state surface is perpendicular to the hyper-plane (a straight line in the two variables case) tangent to the limit state surface in the design point (x_{d1},x_{d2},...x_{dn}). The sensitivity coefficients α represent the influence of the corresponding variable in the failure probabilities.

The following equations hold for the sensitivity coefficients

$$\left|\alpha_i\right| \leq 1; \quad \sum_{1}^{n} \alpha_i^2 = 1 \tag{36}$$

Usually these values α_i, are taken as positives if the correspond to a resistance variable and negatives in the case of effects of action variables.

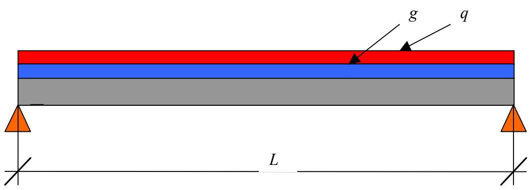
When the limit state surface is highly non-linear, the error of substituting the surface by the tangent hyper-plane in that point can be important. In those cases a minor error is obtained if the limit state surface is substituted by the tangent quadratic surface. That is: taking the square term in the Taylor's series. In this case the method is called Second Order Reliability Method or SORM.

As it is said, the reliability index β and the probability of failure p_f are related by the formula:

$$P_f = \Phi(-\beta); \qquad \beta = \Phi^{-1}(1 - P_f); \tag{37}$$

where $\Phi(\cdot)$ stands for the standard normal distribution and $\Phi^{-1}(\cdot)$ for its inverse.

Example 2
a) Design of a steel beam
Simply supported beam: IPE 240 S235



Span	L = 6,0 m
Cross section area:	A = 39,12 ·10^{-4} m^2
Resistance modulus,	W = 3 243 ·10^{-6} m^3
Yield stress	f_y = 235 MPa

Actions:

Permanent load: g_k = 7,0 kN/m
Variable load: q_k = 3,0 kN/m

Limit State function

$$Z(\boldsymbol{X}) = \theta_1\, W \cdot f_y - \theta_2\,(g + q)\, L^2/8$$

In the following table all the variables are described by the mean, standard deviation and the distribution function, see also [5]:

Variable	Symbol	Mean	Standard deviation	Distribution function
Span	L [m]	6	0	Deterministic
Resistance modulus	W [m^3]	$3\ 243 \cdot 10^{-6}$	0	Deterministic
Resistance model	θ_1 [-]	1	0,1	Lognormal
Yield strength	f_y [MPa]	280	19,6	Lognormal
Action effects model	θ_2 [-]	1	0,2	Lognormal
Permanent load	g [kN/m^2]	0.007	$0.007 \cdot 0{,}1$	Normal
Variable load	q [kN/m^2]	0 .0008	$0.\ 0008 \cdot 0{,}6$	Gumbel

The following results are obtained using the attached MATHEMATICA notebook Form.nb and package Level II.m.

Case a: Results

Reliability index: $\beta =$ 3,82 ; Probability of failure: $\Phi(-\beta) = 6{,}67\times10^{-5}$

Variable	θ_1	fy	θ_2	g	q
Sensitivity coefficient	-0,392	-0,275	0,778	0,304	0,270

From the results can be seen that the reliability index is well adjusted, but also that the limit state is very sensitive to the actions model uncertainty (influence coefficient 0,778).

Case b:

Imagine that we can study more deeply the actions and we arrive to the conclusion that the mean and variances of the actions are the same as before, but now we have reduced the uncertainty model and we have now a coefficient of variation of 10%.

Action effects model	θ_2 [-]	1	0,1	Lognormal

Performing a new study with the new values we obtain:

Reliability index: $\beta =$ 5,04 ; Probability of failure: $\Phi(-\beta) = 2{,}37\times10^{-7}$

Variable	θ_1	f_y	θ_2	g	q
Sensitivity coefficient	-0,505	-0,354	0,505	0,332	0,503

We can see that the influence coefficients are more equilibrated, but now the reliability index is quite high.

Case c:

Perhaps we can reduce the steel profile. We can try with a smaller profile:
New steel profile: IPE 220, W= 252 cm^3

Now the results are the following:

Reliability index: $\beta =$ 3,74; Probability of failure: $\Phi(-\beta) = 9{,}29\times10^{-5}$

Variable	θ_1	f_y	θ_2	g	q
Sensitivity coefficient	-0,524	-0,367	0,524	0,383	0,411

Example 3

Consider the slab of the example in Section 5 (see Figure 8). From the equilibrium consideration was obtained:

$$0{,}8\ fc\ x\ b = As\ fy$$

$$A_s\ f_y(d - 0{,}4\ x) = M$$

Using equilibrium conditions (30) and (31), and considering uncertainty of the models of resistances, θ_1, and actions, θ_2, the following formula for the limit state function can be derived:

$$g(X) = \theta_1 A_s f_y \left(d - \frac{A_s f_y}{2 f_c b} \right) - \theta_2 (g + q) L^2 / 8$$

The following table shows the variables that are considered in the study using the **MATHEMATICA** notebook FORM.nb:

Variable	Symbol	Mean value	Standard deviation	Distribution function
Resistance model	θ_1 [-]	1	0,1	Lognormal
Reinforcement tensile strength	f_y [MPa]	560	30	Lognormal
Reinforcement area	A_s [m^2]	0,00069	0,0000345	Normal
Effective height	d [m]	0,23	0,01	Normal
Concrete compressive strength	f_c [MPa]	30	5,5	Lognormal
Load model	θ_2 [-]	1	0,2	Lognormal
Permanent load	g [kN/m^2]	7	0.7	Normal
Imposed load	q [kN/m^2]	0,8	0,48	Gamma
Span	L [m]	6	-	Deterministic

Results:

Reliability index: $\beta =$ 3,56; Probability of failure: $\Phi(-\beta) = 1{,}87\times10^{-4}$

Variable	θ_1	A_s	f_y	d	f_c	θ_2	g	q
Sensitivity coefficient	-0,383	-0,193	-0,201	-0,177	-0,018	0,761	0,300	0,274

From the results it is possible to draw the following conclusions: the reliability index is a little bit lower; the coefficient of influence of the concrete resistance, f_c, is almost zero, it could be possible to consider it as deterministic without influencing the results; and the uncertainty in the actions has the maximum influence.

Case b:

The new study is performed considering reduced uncertainty in the load model as follows:

Load model	θ_2 [-]	1	0,1	Lognormal

Results:
Reliability index: β = 4,63; Probability of failure: $\Phi(-\beta) = 1{,}82\times10^{-6}$

Variable	θ_1	A_s	f_y	d	f_c	θ_2	g	q
Sensitivity coefficient	-0.500	-0,259	-0,263	-0,235	-0,023	0,500	0,347	0,431

Case c:

Now the reliability index is a bit higher. Reducing the cross section of the reinforcing steel bars:

Reinforcement area	A_s [m^2]	0,00059	0,00059×0,05	Normal
Load model	θ_2 [-]	1	0,1	Lognormal

Results:

Reliability index: β = 3,86; Probability of failure: $\Phi(-\beta) = 5{,}61\times10^{-5}$

Variable	θ_1	A_s	f_y	d	f_c	θ_2	g	q
Sensitivity coefficient	-0,504	-0,260	-0,266	-0,234	-0,020	0,504	0,368	0,401

From both the examples can be seen that reducing the uncertainty in the model of actions, is possible to reduce the steel section without reducing (even increasing) the overall reliability.

7 SYSTEM RELIABILITY

7.1 General

Even in the simplest case of one structural element: a beam or column, more than one ultimate limit state function has to be considered: failures at positive or negative moments or shear of the beam. In most cases, the structure has multiple elements what is called a 'structural system'.

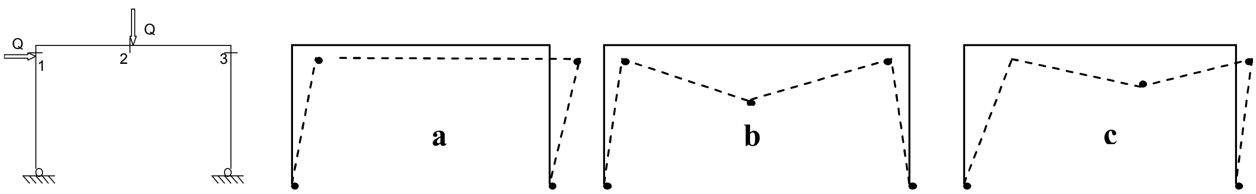
The reliability of the system depends on the reliability of its elements: the effects of actions in each element depend on applied loads, loads and resistances may not be independent, there may be a correlation between the properties of the elements in different parts of the structure. Furthermore, there are limit states for the structure as a whole like the overall deflection or foundation settlement.

When all the different failure modes are identified, a 'fault-tree' or an 'event-tree' of all the failures modes can be established.

Example 4

Consider the simple portal frame of the following Figure ~~4~~ submitted to the horizontal and vertical loads Q. Assuming a plastic behaviour, the frame has three possible modes of collapse. In each path different plastic hinges will be formed:

a) *Sway mode*: Plastic hinges formed at sections 1 and 3;
b) *Beam mode*: Plastic hinges at sections 1, 2 and 3;
c) *Combined mode*: Plastic hinges at sections 2 and 3.



The failure in any path implies the failure of the structure and the event 'structural failure' will be the union of the all n failure modes (n = 3, in the example case). And therefore the probability of failure of the structure will be:

$$P_{\mathrm{f}} = \mathrm{P}(F_S) = P(F_1 \times F_2 \times \ldots \times F_n) \tag{38}$$

where F_i is the event failure in mode i. For each mode m nodes or structural elements have to fail. So the probability of failure of each mode, F_i, will be:

$$P_{\mathrm{f}}(F_i) = \mathrm{P}(F_{1i} \emptyset F_{2i} \emptyset \ldots \emptyset F_{mi}) \tag{39}$$

where F_{ji} is the event failure of the j-th element or node in the i-th failure mode. To reach the collapse of the structure in the mode i, m elements or nodes must fail.

In general the structure is idealized as a parallel system, a series system or a combination of both.

7.2 Parallel systems

In the parallel system are the elements so interconnected that the failure of one or more of the elements does not mean the failure of the whole structure. Such a structure is called a redundant structure. This redundancy can be *active*, when the redundant members can activate before the limit state of any member was reached, or *passive*, when the redundant members only act when a limit state is reached in some member.

It should be taken into account that any hyperstatic structure is not necessary a parallel system: if the elements are brittle, the failure of any of them can lead to a new distribution of stresses such that new failures are reached immediately.

The failure of a pure parallel system with m components is given by:

$$P_{\mathrm{f\,sys}} = \mathrm{P}\left(\bigcap_{1}^{\mathrm{m}} \mathrm{Fj}\right) = \mathrm{P}\left[\bigcap_{1}^{m} (\mathrm{Zj} < 0)\right] \tag{40}$$

where F_j is event of the failure of the j-th component with Z_j limit state function. Thus:

$$\mathrm{P}(F_j) = \mathrm{P}(Z_j < 0) \approx \Phi(-\beta) \tag{41}$$

The probability of failure of the system by FORM is given by:

$$P_{\mathrm{f\,sys}} = \Phi_{\mathrm{m}}(-\boldsymbol{\beta}\,;\boldsymbol{\rho}) \tag{42}$$

where Φ_{m} is the multi-dimensional standard normal distribution, $\boldsymbol{\beta}$ is the vector of the component reliability indices and $\boldsymbol{\rho}$ the $m \times m$ correlation matrix of the reliability indices, given by

$$\rho_{jk} = \sum_i \alpha_{ij}\, \alpha_{ik}\,; \qquad \text{with j, k} = 1, 2, \ldots, \text{m} \tag{43}$$

and α_{ij} is the sensitivity factor of the i-th variable in the j-th component limit state function.

It is a demanding task to calculate the probability of failure of the system. An upper bound of this probability can be obtained as:

$$p_{\mathrm{f\,sys}} = \underset{j,k=1}{\overset{m}{Min}} \left[P(F_j \bigcap F_k) \right] \tag{44}$$

A simple approximation for only two elements:

$$p_{\mathrm{f\,sys}} = \Phi(-\beta_1)\ \Phi(-\beta^*_2); \quad \text{with } \beta^*_2 = (\beta_2 - \rho\,\beta_1) / \sqrt{1-\rho^2} \tag{45}$$

7.3 Series systems

A series system is that the failure of any element leads to the collapse of the structure. It is called a 'weakest link'. Any statically determined structure is a series system. In a similar manner that was shown in the previous section, the failure probability in a pure series system with m components is given by:

$$C = P\left(\bigcup_{1}^{m} \mathrm{Fj} \right) \tag{46}$$

The probability of failure of the system given by FORM is:

$$P_{\mathrm{f\,sys}} = 1 - \Phi_{\mathrm{m}}(\boldsymbol{\beta}\,;\boldsymbol{\rho}) \tag{47}$$

Simple bound to this probability are given by:

$$\underset{1}{\overset{m}{Max}}\left[P(F_j) \right] \le P_{\mathrm{f\,sys}} \le Min\left[\sum_{1}^{m} P(F_j), 1 \right]. \tag{48}$$

These bounds are usually too wide. A method to find more precise bounds can be found in [11].

8 CONCLUDING REMARKS

Elementary methods of structural reliability can be used to assess the reliability of fundamental cases of two random variables when the limit state function is formulated as the difference between the resulting structural resistance and load effect. Software products, usually based on the methods FORM and SORM, must be used in a general case of more basic variables.

Basic principles of the reliability theory provide operational techniques that can be used for estimating the partial factors of basic variables. The assessment of various reliability measures in the new structural design codes is, however, partly based on historical and empirical experiences. Obviously the past experience depends on local conditions including climatic actions and traditionally used construction materials and, consequently, might be in different countries considerably diverse. That is why number of reliability elements and parameters in the present suite of European standards are open for national choice.

MATHCAD sheets, EXCEL sheets and MATHEMATICA notebooks supplement the numerical examples and make it possible to recalculate similar examples of basic structural members exposed to common permanent and variable loads.

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ATTACHMENTS

1. MATHCAD SHEET "GammaRGQ.mcd"

MATHCAD sheet "GammaRGQ.mcd" is intended for calculation of the partial factors γ_R, γ_R and γ_R assuming selected theoretical models.

2. MATHCAD SHEET "Prindex.mcd"

MATHCAD sheet "Prindex.mcd" is intended for determining the reliability index from the failure probability.

3. MATHCAD sheet "RCBeam.mcd"

MATHCAD sheet "RCBeam.mcd" is intended for design of a reinforced concrete beam exposed to permanent and variable loads.

4. MATHCAD sheet "RelRCB.mcd"

MATHCAD sheet "RelRCB.mcd" is intended for reliability analysis of a reinforced concrete beam exposed to permanent and variable loads.

5. EXCEL sheet "RCBeam.xls"

EXCEL sheet "RCBeam.xls" is intended for design of a reinforced concrete beam exposed to permanent and variable loads.

6. MATHEMATICA notebook "Fit_distribution.nb"

MATHEMATICA notebook "Fit_distribution.nb" is intended for fit selected theoretical models to experimental data.

7. MATHEMATICA notebook "FORM.nb"

MATHEMATICA notebook "FORM.nb" is intended for reliability analysis of a structural member.

8. MATLAB package "Level2.m"

MATLAB package "Level2.m" is intended for determining the reliability index using FORM method.

9. MATHCAD sheet "FORM2.mcd"

MATHCAD sheet "FORM2.mcd" is intended for calculation of the reliability index β and failure probability assuming function g(X)= R - E = 0 assuming general three parameter lognormal distribution LN(μ,σ,α) of E and R.

10. MATDCAD sheet "FORM7.mcd"

MATHCAD package "FORM7.mcd" is intended for calculation of the reliability index β and failure probability assuming a non-linear limit state function g(X) and general three parameter lognormal distribution LN(μ,σ,α) for basic variables.

11. EXCEL sheet "FORM7.xls"

MATHCAD package "FORM7.mcd" is intended for calculation of the reliability index β and failure probability assuming a non-linear limit state function g(X) and general three parameter lognormal distribution LN(μ,σ,α) of basic variables

12. MATLAB function "FORM7.m"

MATLAB package "FORM7.m" is intended for determining the probability of failure pf, assuming a pre-defined limit state function of seven basic variables.

13. MATLAB function "Lnden (x, mu, sigma,sk)"

MATLAB function "Lndens" is intended for calculation of the probability density function of three-parameter lognormal distribution. The function is called by the function FORM7 using command “Lndens(ske,me,se)”, and returns the value of probability density function.

15. MATLAB function " Lndist (x, mu, sigma,sk)"

MATLAB function "Lndist" is intended for calculation of the distribution function of three-parameter lognormal distribution. The function is called by the function FORM7 using command “Lndist(skr,mr,sr),” and returns the value of the distribution function.

15. MATLAB function “Ndens (x, mu, sigma)”

MATLAB function "Ndens" evaluates the one-dimensional normal density function. The function is called by the function FORM7 using command “Ndens(x)” (or “Ndens(x,mu,sigma)” or Ndens(x,mu)”), and returns the value of the inverse distribution function.

16. MATLAB function "Ndinv (p, mu, sigma)"

MATLAB function "Ndinv" calculates the inverse distribution function of the normal distribution (determining the reliability index beta). The function is called by the function FORM7 using command “Ndinv(p)” (or “Ndinv(p,mu,sigma)” or Ndinv(p,mu)”), and returns the value of the inverse distribution function.

17. MATLAB function "Ndist (x, mu, sigma)"

MATLAB function "Ndist" evaluates the one-dimensional normal distribution function. The function is called by the function FORM7 using command “Ndist(x)” (or “Ndist(x,mu,sigma)” or Ndist(x,mu)”), and returns the value of the inverse distribution function.

Attachment 1 - MATHCAD sheet GammaRGQ.mcd

GammaR, gammaG and gammaQ

assuming theoretical model

MATHCAD sheet for determination of partial factors γR, γG and γQ.

Study parameters: reliability index and coefficient of variation: $\beta := 0, 0.1 .. 5 \quad V := 0, 0.1 .. 0.5$

1 Coeficients and factors used in EN 1990

Coefficient for 5% fractile	$k := 1.65$	Standardised normal variable	
Coefficient for 0,1 % fractile	$d := 3.09$	Not dicrectly used in this sheet	
Sensitivity factors:	$\alpha R := 0.8$	$\alpha E := -0.7$	FORM factors assumed in EN 1990
Reduced β values:	$\beta R(\beta) := \beta \cdot \alpha R$	$\beta E(\beta) := \beta \cdot \alpha E$	Basic reliability index $\beta = 3.8$

2 Characteristic values (relative values related to the mean, $\xi k = xk / \mu x$)

Normal distribution $\quad \xi kn(V) := (1 - k \cdot V)$ $\quad$ Check: $\quad \xi kn(0.1) = 0.835$

Lognormal distribution

$$\xi kln(V) := \frac{\exp\left[(-k)\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}} \qquad \xi kln(0.1) = 0.844$$

Gumbel distribution

$$\xi kgum(V) := 1 - V \cdot (0.45 + 0.78 \ln(-\ln(0.98))) \qquad \xi kgum(0.35) = 1.908$$

3 Design values (relative values related to the mean, $\xi d = xd / \mu x$)

Normal distribution

$$\xi dn(\beta, V) := (1 - \beta R(\beta) \cdot V) \qquad \xi dn(3.8, 0.1) = 0.696$$

Lognormal distribution

$$\xi dln(\beta, V) := \frac{\exp\left[(-\beta R(\beta))\cdot\sqrt{\ln\left(1+V^2\right)}\right]}{\sqrt{1+V^2}} \qquad \xi dln(3.8, 0.1) = 0.735$$

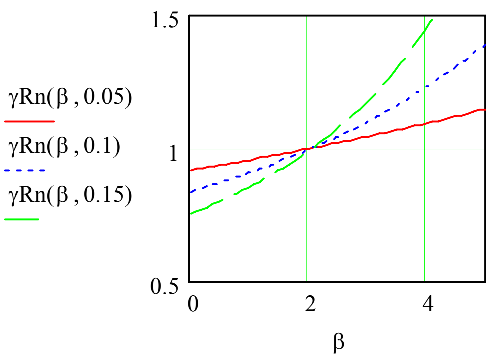
Gumbel distribution

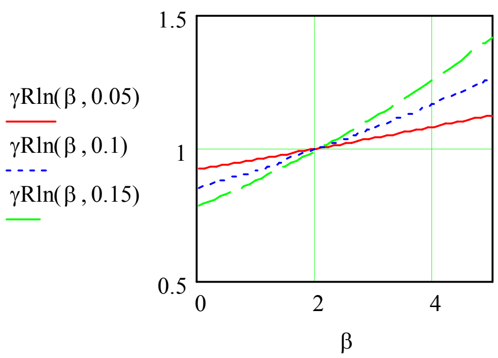
$$\xi dgum(\beta, V) := 1 - V \cdot (0.45 + 0.78 \ln(-\ln(1 - \mathrm{pnorm}(\beta E(\beta), 0, 1))))$$

$$\xi dgum(3.8, 0.1) = 1.387$$

4 GammaR for resistance assuming normal and lognormal distribution

$$\gamma Rn(\beta, V) := \frac{\xi kn(V)}{\xi dn(\beta, V)} \qquad \gamma Rln(\beta, V) := \frac{\xi kln(V)}{\xi dln(\beta, V)} \qquad \gamma Rn(3.8, 0.1) = 1.2$$

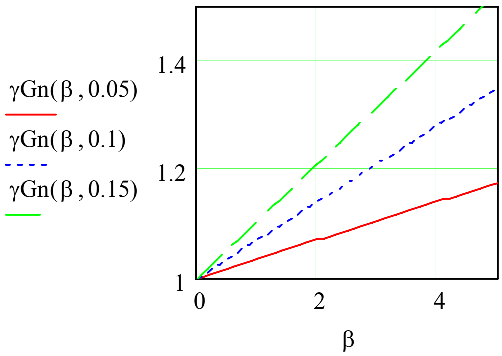


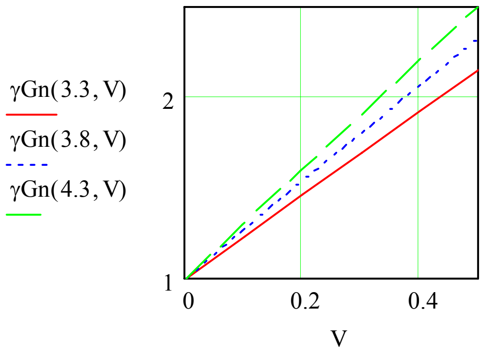


5 GammaG for permanent load assuming normal distribution

$$\gamma Gn(\beta, V) := 1 - \beta E(\beta)\cdot V$$

$$\gamma Rn(3.8, 0.1) = 1.2$$





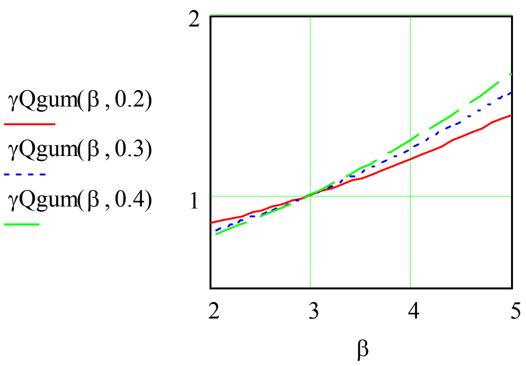
6 GammaQ for variable load assumingGumbel distribution

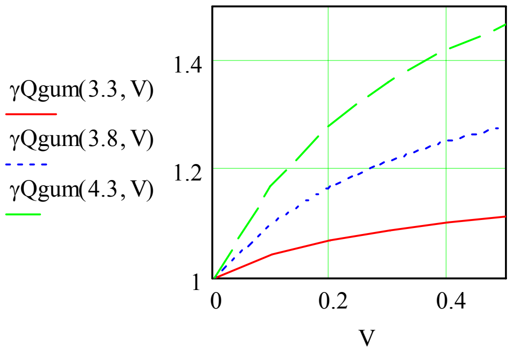
Partial factor γQ:

$$\gamma Qgum(\beta, V) := \left(\frac{\xi dgum(\beta, V)}{\xi kgum(V)}\right)$$

$$\xi dgum(3.8, 0.5) = 2.937$$

$$\gamma Qgum(3.8, 0.5) = 1.279$$





Note. Calculation procedures applied in this sheet for determination of Gamma factors do not take into account model uncertainties of relevant variables. An additional magnifying factor of a magnitude from 1.05 to 1.10 is considered in Eurocode recommendations.

MH, 21.2.2003

Attachment 2 - MATHCAD sheet PrIndex..mcd

Failure probability P and reliability index β

1 Reliability index β for a given failure probability P: β= - $\Phi(P)$

$$P := \begin{pmatrix} 10^{-1} \\ 10^{-2} \\ 10^{-3} \\ 10^{-4} \\ 10^{-5} \\ 10^{-6} \end{pmatrix} \qquad \beta := -\mathrm{qnorm}(P,0,1) \qquad \beta = \begin{pmatrix} 1.28 \\ 2.33 \\ 3.09 \\ 3.72 \\ 4.26 \\ 4.75 \end{pmatrix}$$

2 Failure probability P *for a given* reliability index β: $P = \Phi^{-1}(\beta)$

$$\beta := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} \qquad P := \mathrm{pnorm}(-\beta,0,1) \qquad P = \begin{pmatrix} 0.16 \\ 0.02 \\ 1.35 \times 10^{-3} \\ 3.17 \times 10^{-5} \\ 2.87 \times 10^{-7} \\ 9.87 \times 10^{-10} \end{pmatrix}$$

3 Failure probability P_1 (e.g. one year) and P_n : P_n= 1 - (1 - $P_1)^n$

$$P1 := \begin{pmatrix} 7.23 \times 10^{-5} \\ 1.33 \times 10^{-5} \\ 1.3 \times 10^{-6} \end{pmatrix} \quad n := 50 \quad Pn := 1-(1-P1)^n \quad Pn = \begin{pmatrix} 3.61 \times 10^{-3} \\ 6.65 \times 10^{-4} \\ 6.5 \times 10^{-5} \end{pmatrix}$$

4 Reliability index β for one and 50 years $\Phi(\beta_n) = \Phi(\beta_1)^n$

Directly from β_1:

$$\beta 1 := \begin{pmatrix} 3.8 \\ 4.2 \\ 4.7 \end{pmatrix} \qquad \beta n := \mathrm{qnorm}\left(\mathrm{pnorm}(\beta 1,0,1)^n,0,1\right) \qquad \beta n = \begin{pmatrix} 2.69 \\ 3.21 \\ 3.83 \end{pmatrix}$$

Alternatively through the probability P_n:

$$P1 := \mathrm{pnorm}(-\beta 1,0,1) \quad Pn := 1-(1-P1)^n \quad \beta n := -\mathrm{qnorm}(Pn,0,1) \qquad \beta n = \begin{pmatrix} 2.69 \\ 3.21 \\ 3.83 \end{pmatrix}$$

Attachment 3 - MATHCAD sheet RCBeam.mcd

Reinforced Concrete Design: Slab or Beam

Partial factor method, global factor method and permissible stresses.

A simply supported reinforced concrete slab or beam of the span L is exposed to permanent load g (including self weight) and imposed load q.
Units: m, kN, kNm, kN/m and MPa for strength

1 Load effect

Span of a simply supported beam **Define:** $L := 6$

Define: Permanent load: $g := 7$ Imposed load: $q := 3$

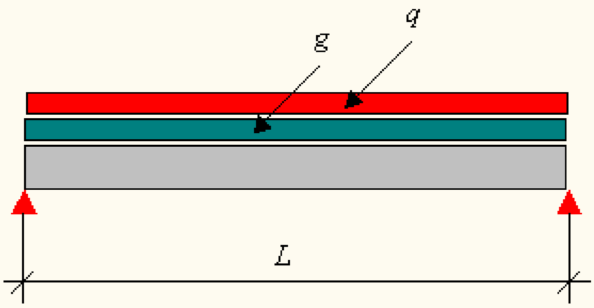
Load Factors to be used

Partial factor method use: $\gamma G = 1.35$, $\gamma Q = 1.5$ and $\gamma = 1.0$
Global safety factor use: $\gamma G = 1.0$, $\gamma Q = 1.0$ and $\gamma = 1.9$
Permissible stresses: $\gamma G = 1.0$, $\gamma Q = 1.0$ and $\gamma = 1.0$

Bending moment

Define: $\gamma G := 1.0$ $\gamma Q := 1.0$ $\gamma := 1.9$

$$ME := (g \cdot \gamma G + q \cdot \gamma Q) \cdot \frac{L^2}{8} \qquad ME = 45$$



2 Cross section of a slab or beam

Rectangular cross section

h = L/20 to L/25, thus
h ~ 0,25 m, d ~ 0,22 m

Define: $b := 1$ $d := 0.22$

3 Reinforcement area according to partial factors and global factor method

Material properties **Define:** $fck := 20$ $fyk := 500$

Materials Factors to be used

Partial factor method use: $\gamma c = 1.5$, $\gamma s = 1.15$
Global safety factor use: $\gamma c = 1.0$, $\gamma s = 1.0$

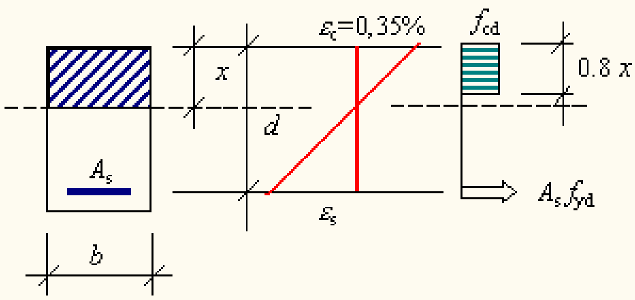
Define: $\gamma c := 1.0$ $\gamma s := 1.00$

Design properties: $fcd := \frac{fck}{\gamma c}$ $fyd := \frac{fyk}{\gamma s}$

Reinforcement area:

$$As := \frac{fcd \cdot b \cdot d \cdot \left(1 - \sqrt{1 - 2 \cdot \frac{0.001 ME \cdot \gamma}{fcd \cdot b \cdot d^2}}\right)}{fyd}$$

$$As = 8.15 \times 10^{-4}$$



Check of reinforcement ratio [%] ρ > 0.13

$$\rho := \frac{100 As}{b \cdot d} \qquad \rho = 0.37 \qquad check1 := if(\rho > 0.13, 1, 0) \quad check1 = 1$$

Check of the neutral axis x/d = ξ < 0.45 (for the balanced section ξ = 0.617d)

$$\xi := 0.01 \cdot \rho \cdot \frac{fyd}{0.8 \cdot fcd} \qquad \xi = 0.116 \qquad check2 := if(\xi < 0.45, 1, 0) \quad check2 = 1$$

4 Reinforcement area according to Permissible stresses method
fcp = 5,5, fyp = 275 (CP 114)

Bending moment in accordance with permissible stress method $M := (g+q)\cdot\frac{L^2}{8}$ $M = 45$

Reinforcement area As for the effective depth d = 0,22 m

Neutral axis x from the condition Atot * x = S, where Atot = b*(x+t), S = b * (x² / 2 + t * d):

Finding A from the condition of permissible stress in conctrete νc=fcp, i.e. M/(0.5 r b x) = 5.5 MPa

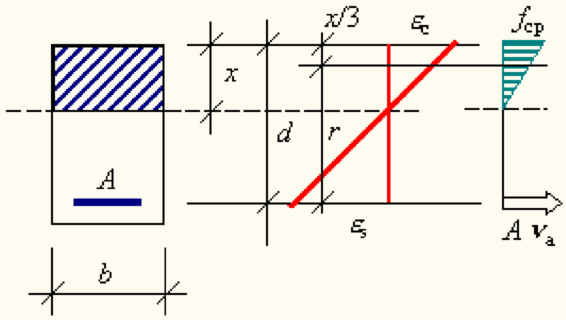
Guess value: $A := 0.00100$

Given $$5.5 = \frac{2\cdot 0.001\cdot M}{\left[d - \dfrac{15\cdot\frac{A}{b}\cdot\left(\sqrt{2\cdot\dfrac{d}{15\cdot\frac{A}{b}}}-1\right)}{3}\right]\cdot b\cdot\left[15\cdot\frac{A}{b}\cdot\left(\sqrt{2\cdot\dfrac{d}{15\cdot\frac{A}{b}}}-1\right)\right]}$$

$A := \text{Find}(A)$ $A = 2.04\times10^{-3}$

Check: $t := 15\cdot\frac{A}{b}$ $t = 0.031$

$x := t\cdot\left(\sqrt{2\cdot\frac{d}{t}}-1\right)$ $r := d - \frac{x}{3}$



$va := \frac{0.001M}{r\cdot A}$ $va = 115.15$

$vc := \frac{2\cdot 0.001M}{r\cdot b\cdot x}$ $vc = 5.5$

$Nc := vc\cdot b\cdot x\cdot 0.5$ $Nc = 0.235$ $Na := A\cdot va$ $Na = 0.235$

Reinforcement ratio [%] $\rho := \frac{100A}{b\cdot d}$ $\rho = 0.927$

Design of ideally reinforced cross section

Choosing fy $fc := 5.5$ $fy := 275$

$\lambda := \frac{15\cdot fc}{fy + 15\cdot fc}$ $\delta := 1 - 0.3333\cdot\lambda$ $\alpha := \sqrt{\frac{2}{\lambda\cdot\delta\cdot fc}}$ $\alpha = 1.307$

$d := \alpha\cdot\sqrt{\frac{0.001M}{b}}$ $d = 0.277$ $\beta := \frac{1}{\alpha\cdot\delta\cdot fy}$ $\delta = 0.923$ $\alpha = 1.307$ $\beta = 3.015\times10^{-3}$

$A := \beta\cdot\sqrt{0.001M}$ $A = 6.396\times10^{-4}$

Check $vb := \frac{2\times 0.001M}{b\cdot d^2\cdot\lambda\cdot\delta}$ $vb = 5.5$ $va := \frac{0.001M}{A\cdot d\cdot\delta}$ $va = 275$

$Nb := fc\cdot b\cdot\lambda\cdot d\cdot 0.5$ $Nb = 0.176$ $Na := A\cdot fy$ $Na = 0.176$ $\rho := \frac{100A}{b\cdot d}$ $\rho = 0.231$

MH 26.2.2003

Attachment 4. MATHCAD sheet RelRCB.mcd

RelRCB - Reliability analysis of Reinforced Concrete slab or Beam

Design, reliability, parametric study of γG, γQ. Applied units: m, MN, MPa

Characteristic actions: $Gk := 0.007$ $Qk := 0.003$ **Material strength:** $fk := 500$ $ck := 20$

Material factors: $\gamma c := 1.5$ $\gamma m := 1.15$ For the global factor method $\gamma c = \gamma m = 1.0$

Load factors: Prameters: $\gamma G := 1, 1.05 .. 1.5$ $\gamma Q := 1, 1.05 .. 1.6$ Global factor if any $s0 := 1.0$

Load effect: $L := 6$ $ek := \frac{L^2}{8}$ $ME(\gamma G, \gamma Q) := s0 \cdot ek \cdot (\gamma G \cdot Gk + \gamma Q \cdot Qk)$ $ME(1.35, 1.5) = 0.063$

Cross-section: $b := 1$ $d := 0.25 - 0.03$ Choose reinforcement ? e.g. $\boxed{A(\gamma G, \gamma Q) := 0.00204}$

If not, than A = $A(\gamma G, \gamma Q) := b \cdot d \cdot \left(1 - \sqrt{1 - \frac{\gamma c \cdot 2 \cdot ME(\gamma G, \gamma Q)}{ck \cdot b \cdot d^2}}\right) \frac{ck \cdot \gamma m}{fk \cdot \gamma c}$

If yes, put the red box on this line> $\rho(\gamma G, \gamma Q) := \frac{100\, A(\gamma G, \gamma Q)}{d}$

Reinforcement area [cm²] $10000\, A(1.35, 1.5) = 6.917$ Reinforcement ratio [%] $\rho(1.35, 1.5) = 0.314$

Checking resistance Internal arm: $zk(\gamma G, \gamma Q) := d - \frac{A(\gamma G, \gamma Q) \cdot fk \cdot \gamma c}{2 \cdot ck \cdot \gamma m}$ $zk(1.2, 1.4) = 0.21$

Resistance moment $MR(\gamma G, \gamma Q) := A(\gamma G, \gamma Q) \cdot zk(\gamma G, \gamma Q) \cdot \frac{fk}{\gamma m}$ $MR(1.35, 1.5) = 0.063$

Reliability analysis, of the limit state function Z=R-E=r*A*f*z-e*(G+Q)

Resistance variables:

Model parameter $\mu r := 1$ $wr := 0.1$ $\sigma r := wr \cdot \mu r$

Steel area $\mu A(\gamma G, \gamma Q) := A(\gamma G, \gamma Q)$ $wA := 0.05$ $\sigma A(\gamma G, \gamma Q) := wA \cdot \mu A(\gamma G, \gamma Q)$

Steel strength $\mu f := 560$ $wf := 0.05356$ $\sigma f := wf \cdot \mu f$ $\sigma f = 29.994$

Internal arm $\mu z(\gamma G, \gamma Q) := zk(\gamma G, \gamma Q)$ $wz := 0.07$ $\sigma z(\gamma G, \gamma Q) := wz \cdot \mu z(\gamma G, \gamma Q)$ $\sigma z(1.2, 1.4) = 0.015$

Not directly needed Concrete area $\mu d := d$ $wd := 0.03$ $\sigma d := wd \cdot \mu d$

Concrete strenght $\mu c := 30$ $wc := 0.167$ $\sigma c := wc \cdot \mu c$

Action variables:

Model parameter - Lognorm.d. $\mu e := ek$ $we := 0.1$ $\sigma e := we \cdot \mu e$ $\alpha e := 3we + we^3$

Permanent Load - normal d. $\mu G := Gk$ $wG := 0.1$ $\sigma G := wG \cdot \mu G$

Variable Load - Gumbel d. $\mu Q := 0.0008$ $wQ := 0.6$ $\sigma Q := wQ \cdot \mu Q$ $\alpha Q := 1.14$

Parameters of R $\mu R(\gamma G, \gamma Q) := \mu r \cdot \mu A(\gamma G, \gamma Q) \cdot \mu f \cdot \mu z(\gamma G, \gamma Q)$ $\mu R(1.35, 1.5) = 0.081$

$$\sigma R(\gamma G, \gamma Q) := \mu R(\gamma G, \gamma Q) \cdot \begin{pmatrix} wr^2 + wA^2 + wf^2 + wz^2 \ldots \\ + wr^2 \cdot wA^2 + wr^2 \cdot wf^2 + wr^2 \cdot wz^2 \ldots \\ + wA^2 \cdot wf^2 + wA^2 \cdot wz^2 + wf^2 \cdot wz^2 \ldots \\ + wr^2 \cdot wA^2 \cdot wf^2 + wr^2 \cdot wf^2 \cdot wz^2 \ldots \\ + wA^2 \cdot wf^2 \cdot wz^2 + wA^2 \cdot wf^2 \cdot wz^2 \ldots \\ + wr^2 \cdot wA^2 \cdot wf^2 \cdot wz^2 \end{pmatrix}^{0.5} \qquad \sigma R(1.35, 1.5) = 0.012$$

Parameters of E $\mu E := \mu e \cdot (\mu G + \mu Q)$ $\mu E = 0.035$

$$\sigma E := \mu E \cdot \sqrt[2]{\left[we^2 + \frac{\left(\sigma G^2 + \sigma Q^2\right)}{(\mu G + \mu Q)^2} + we^2 \cdot \frac{\left(\sigma G^2 + \sigma Q^2\right)}{(\mu G + \mu Q)^2}\right]} \qquad wE := \frac{\sigma E}{\mu E} \qquad \sigma E = 5.201 \times 10^{-3}$$

Skewness of E:

$$\alpha E := \frac{(\mu E)^3 \cdot \left[\alpha e \cdot we^3 + \frac{\left(\alpha Q \cdot \sigma Q^3\right)}{\left(\sigma G^2 + \sigma Q^2\right)^{1.5}} \cdot \left[\frac{\left(\sigma G^2 + \sigma Q^2\right)}{(\mu G + \mu Q)^2} \right]^{1.5} + 6 \cdot we^2 \cdot \frac{\left(\sigma G^2 + \sigma Q^2\right)}{(\mu G + \mu Q)^2} \right]}{\sigma E^3} \qquad wE = 0.148 \qquad \alpha E = 0.392$$

Parameter C:

$$C := 2^{\frac{-1}{3}} \cdot \left(\sqrt{\alpha E^2 + 4} + \alpha E\right)^{\frac{1}{3}} - 2^{\frac{-1}{3}} \cdot \left(\sqrt{\alpha E^2 + 4} - \alpha E\right)^{\frac{1}{3}} \qquad C = 0.13$$

Parameters of transformed variable:

$$mE := -\ln(|C|) + \ln(\sigma E) - (0.5) \cdot \ln\left(1 + C^2\right) \quad sE := \sqrt{\ln\left(1 + C^2\right)} \quad x0 := \mu E - \frac{1}{C}\sigma E \quad x0 = -4.883 \times 10^{-3}$$

Probability density of E for positive αE>0 approximated by three parameter lognormal distribution:

$$Eln(x) := dlnorm(x - x0, mE, sE)$$

First estimate β0 assuming normal distribution of E, R, and margin Z = R - E

Parameters of Z

$$\mu Z(\gamma G, \gamma Q) := \mu R(\gamma G, \gamma Q) - \mu E \qquad \mu Z(1.2, 1.4) = 0.038$$

$$\sigma Z(\gamma G, \gamma Q) := \sqrt[2]{\sigma R(\gamma G, \gamma Q)^2 + \sigma E^2} \qquad \sigma Z(1.2, 1.4) = 0.012$$

Reliability index β

$$\beta 0(\gamma G, \gamma Q) := \left| \frac{\mu Z(\gamma G, \gamma Q)}{\sigma Z(\gamma G, \gamma Q)} \right| \quad pf(\gamma G, \gamma Q) := pnorm(-\beta 0(\gamma G, \gamma Q), 0, 1) \quad \beta 0(1.35, 1.5) = 3.612$$

Index β1 assuming Gamma distribution of E and lognormal of R

Gamma distribution of E :

Gamma distribution of E:

$$k := \left(\frac{\mu E}{\sigma E}\right)^2 \qquad \lambda := \left(\frac{\mu E}{\sigma E^2}\right) \qquad E(x) := dgamma(\lambda \cdot x, k) \cdot \lambda$$

Lognormal distribution of R having the lower limit 0

$$C(\gamma G, \gamma Q) := \frac{\sigma R(\gamma G, \gamma Q)}{\mu R(\gamma G, \gamma Q)} \quad m(\gamma G, \gamma Q) := \ln(\sigma R(\gamma G, \gamma Q)) - \ln(|C(\gamma G, \gamma Q)|) - (0.5) \cdot \ln\left(1 + C(\gamma G, \gamma Q)^2\right)$$

$$s(\gamma G, \gamma Q) := \sqrt{\ln\left(1 + C(\gamma G, \gamma Q)^2\right)} \qquad Rln(x, \gamma G, \gamma Q) := plnorm(x, m(\gamma G, \gamma Q), s(\gamma G, \gamma Q))$$

Probability and reliability index β: E Gamma

$$pf(\gamma G, \gamma Q) := \int_0^{1 \cdot \mu R(\gamma G, \gamma Q)} E(x)\, Rln(x, \gamma G, \gamma Q)\, dx \qquad \beta 1(\gamma G, \gamma Q) := -qnorm(pf(\gamma G, \gamma Q), 0, 1)$$

$$\beta 1(1.35, 1.5) = 4.195$$

Index β2 assuming three parameter lognormal of E and lognormal of R

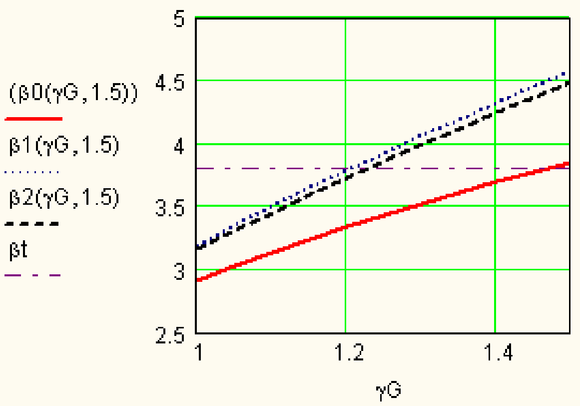
$$pf(\gamma G, \gamma Q) := \int_0^{1 \cdot \mu R(\gamma G, \gamma Q)} Eln(x)\, Rln(x, \gamma G, \gamma Q)\, dx \qquad \beta 2(\gamma G, \gamma Q) := -qnorm(pf(\gamma G, \gamma Q), 0, 1)$$

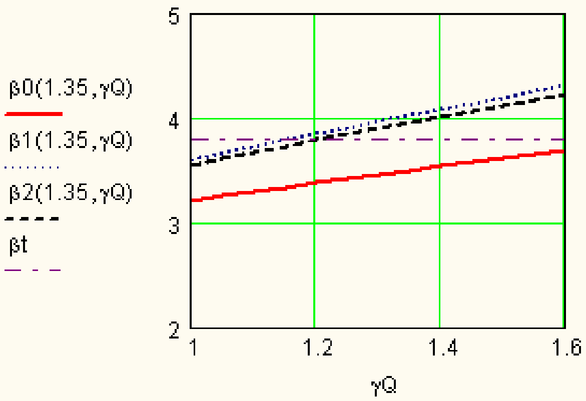
$$\beta 2(1.35, 1.5) = 4.118$$

Parametric study of γG and γQ

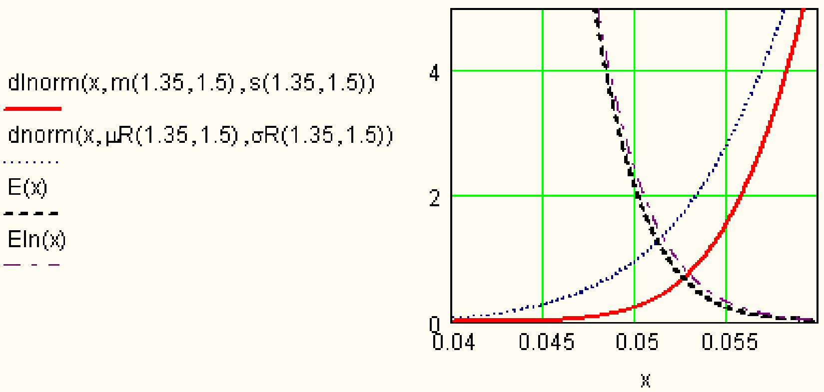
Target reliability index: βt := 3.8

Reliability index
β0 - E normal - R normal
β1 - E Gamma - R Lognormal
β2 - E three parameter lognormal - R lognormal

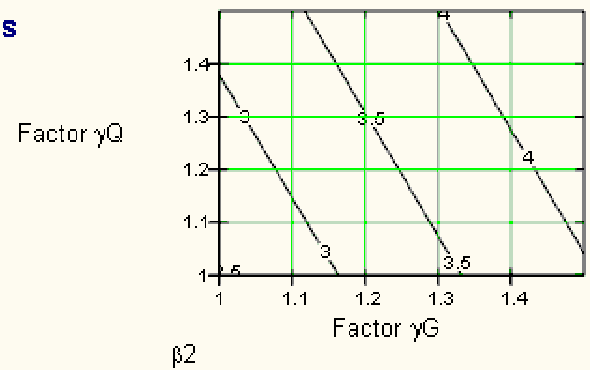




Probability density functions



Contour plotting of β2 versus partial factors γG and γQ



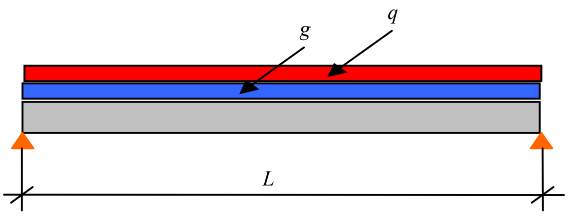
Attachment 5. EXCEL sheet RCBEAM.XLS

REINFORCED CONCRETE BEAM OR SLAB

Partial factor or global safety factor only

Bending moment

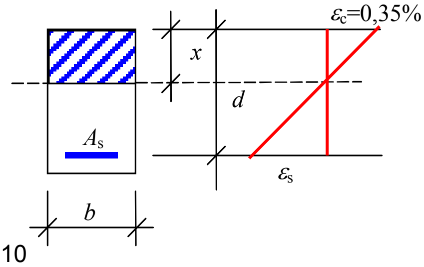
L[m]= 6,00
gk[kN/m]= 7,00 gammaG= 1,35
qk[kN/m]= 3,00 gammaQ= 1,5
ME[kNm]= 62,78



Rectangular cross-section

Dimensions b= 1 d= 0,17
Factors γc= 1,5 αcc= 1 γs= 1,15
Concrete fck [MPa] = 20 fcd=acc*fck/γc= 13,3 fctm= 2,2
Rebars fyk [MPa] = 500 fyd=fyk/γσ= 434,8
Reratio ρmin [%] = 0,130
ξ=x/d< ξmax 0,45

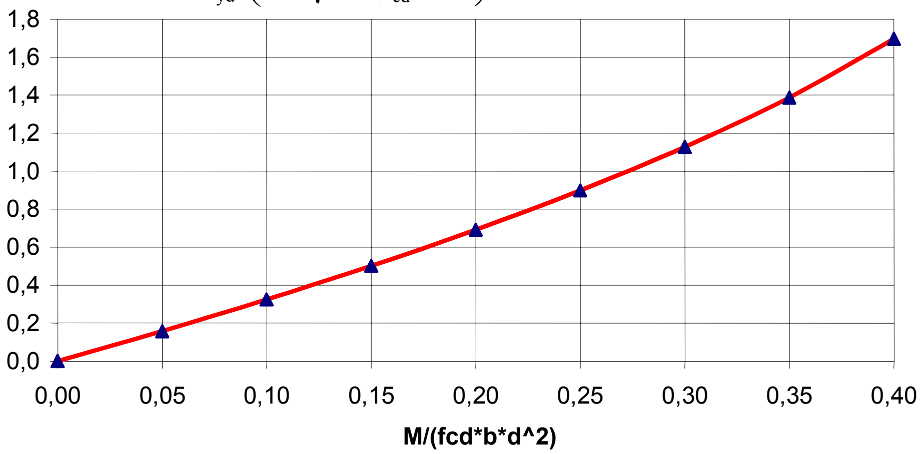
Estimate z ~ 0,9 d
As~Md/(z*fyd)= 0,000944
A[m^2]= 0,000933
ρ [%] = 0,55
ρ >ρmin ? PRAVDA
ξ=x/d= 0,22
ξ<ξmax? PRAVDA ρ max[%] = 1,10



General Table

m = Md/ /(b*d^2*fcd)	Reratio ρ [%]	ξ=x/d	z/d = 1 - 0,4*ξ	εs [%]	As m^2	Md [kNm]
0,00	0,00000	0,00	1,00	-	0,00000	0,0
0,05	0,15737	0,06	0,97	5,11	0,00027	19,3
0,10	0,32376	0,13	0,95	2,30	0,00055	38,5
0,15	0,50091	0,20	0,92	1,36	0,00085	57,8
0,20	0,69124	0,28	0,89	0,89	0,00118	77,1
0,25	0,89821	0,37	0,85	0,61	0,00153	96,3
0,30	1,12714	0,46	0,82	0,41	0,00192	115,6
0,35	1,38698	0,57	0,77	0,27	0,00236	134,9
0,40	1,69521	0,69	0,72	0,16	0,00288	154,1

$$\rho = \frac{A_s}{bd} = \frac{f_{cd}}{f_{yd}}\left(1-\sqrt{1-\frac{2M_E}{f_{cd}bd^2}}\right),\ \rho_{min} \approx 0{,}0013,\ \rho_{max} \approx 0{,}011$$

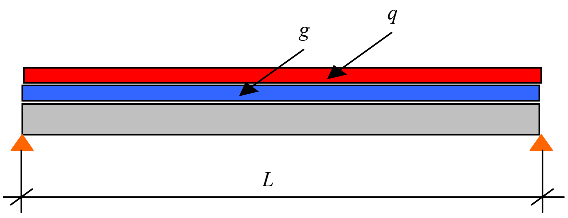


REINFORCED CONCRETE BEAM OR SLAB

Partil factor or global safety factor method

Bending moment

L[m]=	6.00		
gk[kN/m]=	7.00	gammaG=	1.35
qk[kN/m]=	3.00	gammaQ=	1.5
ME[kNm]=	62.78		



Rectangular cross-section

	Dimensions	b=	1	d=	0.17
Factors	γc= 1.5	αcc=	1	γs=	1.15
Concrete	fck [MPa] = 20	ıcc*fck/γc=	13.3	fctm=	2.2
Rebars	fyk [MPa] = 500	ƴd=fyk/γσ=	434.8		
Reratio	ρmin [%] =	0.130			
ξ=x/d<	ξmax	0.45			

Estimate z ~ 0,9 d

As~Md/(z*fyd)= 0.000944

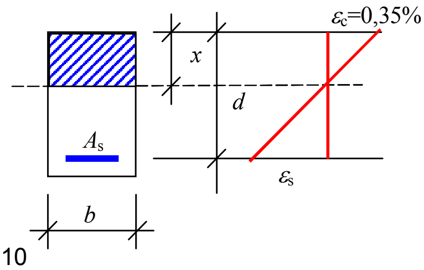
A[m^2]= 0.000933

ρ [%] = 0.55

ρ >ρmin ? PRAVDA

ξ=x/d= 0.22

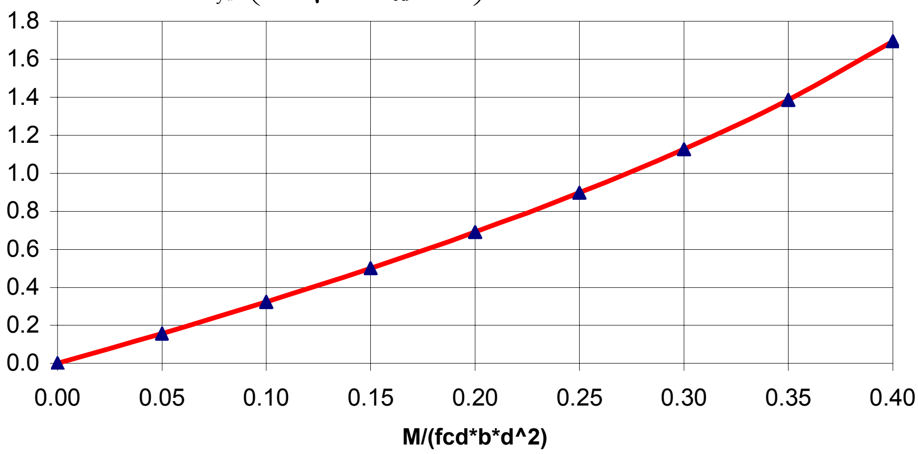
ξ<ξmax? PRAVDA › max[%] = 1.10



General Table

m = Md/ /(b*d^2*fcd)	Reratio ρ [%]	ξ=x/d	'd = 1 - 0,4'	εs [%]	As m^2	Md [kNm]
0.00	0.00000	0.00	1.00	-	0.00000	0.0
0.05	0.15737	0.06	0.97	5.11	0.00027	19.3
0.10	0.32376	0.13	0.95	2.30	0.00055	38.5
0.15	0.50091	0.20	0.92	1.36	0.00085	57.8
0.20	0.69124	0.28	0.89	0.89	0.00118	77.1
0.25	0.89821	0.37	0.85	0.61	0.00153	96.3
0.30	1.12714	0.46	0.82	0.41	0.00192	115.6
0.35	1.38698	0.57	0.77	0.27	0.00236	134.9
0.40	1.69521	0.69	0.72	0.16	0.00288	154.1

$$\rho = \frac{A_s}{bd} = \frac{f_{cd}}{f_{yd}}\left(1-\sqrt{1-\frac{2M_E}{f_{cd}bd^2}}\right),\ \rho_{\min} \approx 0{,}0013,\ \rho_{\max} \approx 0{,}011$$



Attachment 6 – MATHEMATICA notebook `Fit_distribution.nb`

Fit data to distribution functions

Statistical characterization of data. It takes the following steps:

1- Obtain the characteristics of datas: number of elements, mean, variance, standard deviation and coefficeint of variation

2- By the method of moments obtain the indicated distribution parameters

3-Manually the values of the steps in the hystogram are introduced, and then perform the Chi square test of goodness of fit and draw the hystogram and the adjusted density distribution function. Gives two values of Confidence level, the first one corresponding to a degrees of freedom equal number of intervals minus 1 and the second one number of intervals -3.

4-Draw the data points asignig a probability of $p = m/(n+1)$. m = number of order, n= number of data and the cumulative distribution function, both in real scale and double logarithmic scale (i.e., z = -Log(-Log x))

Distributions defined: normal or gauss, lognormal or lognor, gamma, weibull, and extrems or gumbel

Last updated 2 May 2004

Data

Packages needed

```
$TextStyle = {FontSize -> 14, FontFamily -> "Arial"};
```

Data

```
data = {1.3, 4.3, 1.3, 5.4, 3.7, 3.8, 4., 2.9, 3.2, 4.5, 4., 3.4, 2.4,
1.8, 1.7, 2.2, 4.1, 2.6, 4.1, 3.3, 3.5, 3.7, 2.4, 2.8, 2.5, 3., 3.3, 2.6,
2.9, 2.4, 2.6, 2.9, 2.8, 3.1, 3.2, 3.4, 3.5, 3.2, 3.3, 3.};
distributions = {"normal", "lognormal", "gamma"};
numdistributions = Length[distributions];
```

Descriptive statistic

```
ndata = Length[data]
mean = Mean[data]
variance = Variance[data]
deviation = StandardDeviation[data]
coefvariation = deviation / mean
40
3.1025
0.718199
0.847466
0.273156
```

Distribution functions

Parameters of distribution

```
DoBIfBdistributionsQiU >>> "normal" ØØ distributionsQiU >>> "gauss",
                par1 > mean;
                par2 > deviation;
                distAiE > NormalDistributionApar1, par2E,
          IfBdistributionsQiU >>> "lognormal" ØØ distributionsQiU >>> "lognor",
                      mean² , deviation²
                w > ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢;
                          mean²
                             mean²
                par1 > .5 LogB ¢¢¢¢¢¢¢¢F;
                                w
                       R
                par2 >   LogAwE ;
                distAiE > LogNormalDistributionApar1, par2E,
          IfBdistributionsQiU >>> "gamma",
                         mean²
                par1 > ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢;
                       deviation²
                       deviation²
                par2 > ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢;
                          mean
                distAiE > GammaDistributionApar1, par2E,
          IfBdistributionsQiU >>> "weibull",
                par1 > a ®.
                                mean²                          GammaA1 , 1/a¢E
                    FindRootB ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ >> ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢, 9a, .1=
                              deviation²        GammaA1 , 2/a¢E . GammaA1 , 1/a¢E²
                             mean
                par2 > ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢;
                       GammaB1 , 1/par1¢F
                distAiE > WeibullDistributionApar1, par2E,
          IfBdistributionsQiU >>> "extrems" ØØ distribucionesQiU >>> "gumbel",
                                       R
                                        6 EulerGamma deviation
                par1 > media . NB ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢F;
                                           q
                                          R
                              deviation     6
                par2 > NB ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢F;
                                    q
                distAiE > ExtremeValueDistributionApar1, par2E,
          PrintA"Error: the distribution ", i, " I", ditributionsQiU, "is not defined
          BreakAEFFFFF, 9i, numdistributions=F
```

Quantiles

```
DoA
    PrintAdistAiEE;
    PrintA    NAQuantileAdistAiE, 0.5E, 4E, " ",
         NAQuantileAdistAiE, 0.75E, 4E, " ",
         NAQuantileAdistAiE, 0.9E, 4E, " ",
         NAQuantileAdistAiE, 0.95E, 4E, " ",
         NAQuantileAdistAiE, 0.99E, 4E, " ",
         NAQuantileAdistAiE, 0.999E, 4E, " ",
         NAQuantileAdistAiE, 0.9999E, 4EE;
     PrintA" "E,
     9i, 1, numdistributions=E
```

```
NormalDistributionA3.1025, 0.847466E

3.1025 3.67411 4.18857 4.49646 5.074 5.72137 6.25424


LogNormalDistributionA1.09623, 0.268257E

2.99285 3.58645 4.22075 4.65281 5.58612 6.85654 8.11633
```

```
GammaDistributionA13.4023, 0.23149E
3.02569 3.62454 4.226 4.61518 5.4059 6.38894 7.27698
```

Goodness of Fit Test

```
xmax > 1.5 MaxAdataE;
xmin > .9 MinAdataE;
```

Defining the intervals

Adjust the intervals length in order to obtain more or less the same number of samples in each interval (minimum samples)

```
intervals > 92.3, 2.7, 3, 3.3, 3.7, 4.1, xmax=;
count > RangeCountsAdata, intervalsE
95, 7, 5, 6, 7, 5, 5, 0=
frecuencies > DropAcount, .1E;
numintervals > LengthAintervalsE;
PrependToAintervals, xminE;
total > Plus // frecuencies;
hystogr > 99intervalsQ1U, 0.==;
```

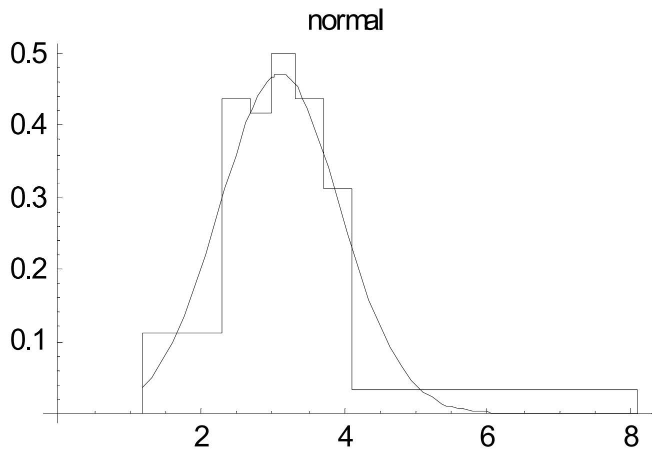
Draws the histogram

```
                                              frecuenciesQiU
DoBAppendToBhystogr, :intervalsQiU, ÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇ>F;
                                    total IintervalsQi , 1U . intervalsQiUM
                                                 frecuenciesQiU
    AppendToBhystogr, :intervalsQi , 1U, ÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇÇ>F,
                                       total IintervalsQi , 1U . intervalsQiUM
    9i, numintervals=F
AppendToAhystogr, 9intervalsQnumintervals , 1U, 0.=E;
hystogram > ListPlotAhystogr, PlotJoined Â True, DisplayFunction Â IdentityE;
```

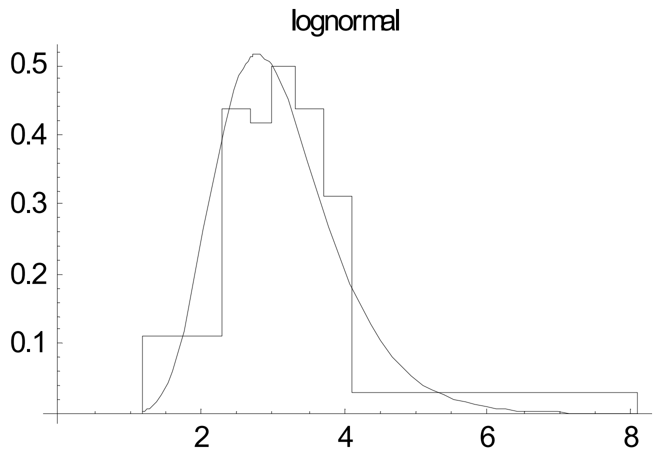
Performs of Chi square test and draw the density function

```
DoBestadW > 0.;
        DoBnj > frecuenciesQjU;
            ej > ICDFAdistAiE, intervalsQj , 1UE . CDFAdistAiE, intervalsQjUEM total;
                                 Inj . ejM2
            estadW > estadW , ÇÇÇÇÇÇÇÇÇÇÇÇ,
                                     ej
        9j, numintervals=F;
        alfa > I1 . CDFAChiSquareDistributionAnumintervals . 1E, estadWEM 100.;
        alfa1 > I1 . CDFAChiSquareDistributionAnumintervals . 3E, estadWEM 100.;
        PrintAdistAiEE;
        PrintA"Estimator W > ", estadW, "    Degrees of freedom > ", numintervals . 1E;
        PrintA"Confidence level between ", alfa1, " and ", alfa, " %"E;
        PrintA" "E;
        fdensity > PlotAPDFAdistAiE, xE, 9x, intervalsQ1U , .001, intervalsQnumintervals ,
                        DisplayFunction Â IdentityE;
        ShowA9hystogram , fdensity=,
            PlotLabel Â distributionsAAiEE,
            DisplayFunction Â $DisplayFunctionE,
    9i, numdistributions=F;
```

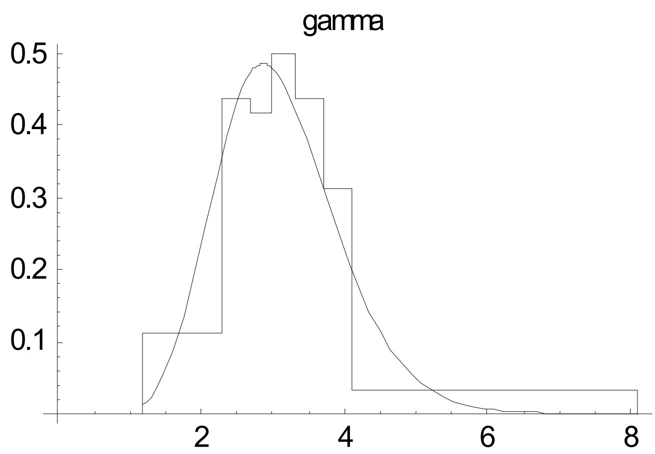
```
NormalDistributionA3.1025, 0.847466E
Estimator W > 0.635298    Degrees of freedom > 6
Confidence level between 95.9062 and 99.5782 %
```




```
LogNormalDistributionA1.09623, 0.268257E
Estimator W > 1.31828    Degrees of freedom > 6
Confidence level between 85.8267 and 97.064 %
```




```
GammaDistributionA13.4023, 0.23149E
Estimator W > 1.00414    Degrees of freedom > 6
Confidence level between 90.9168 and 98.5455 %
```

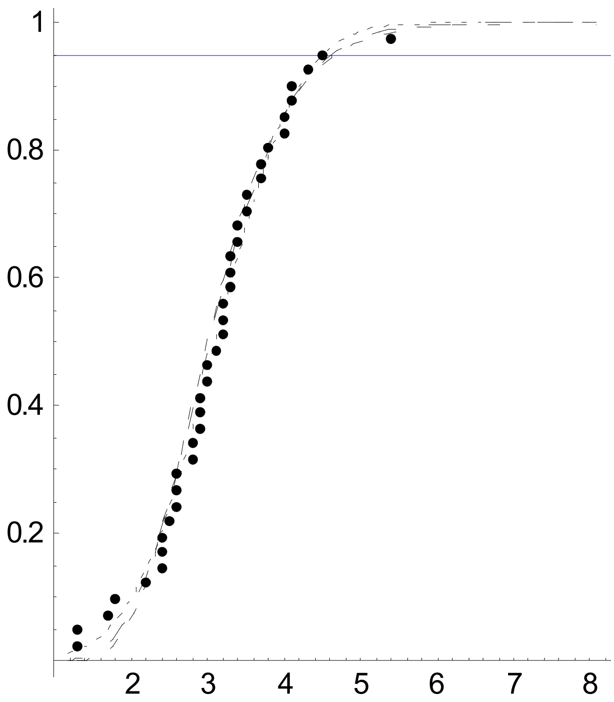


Distribution functions

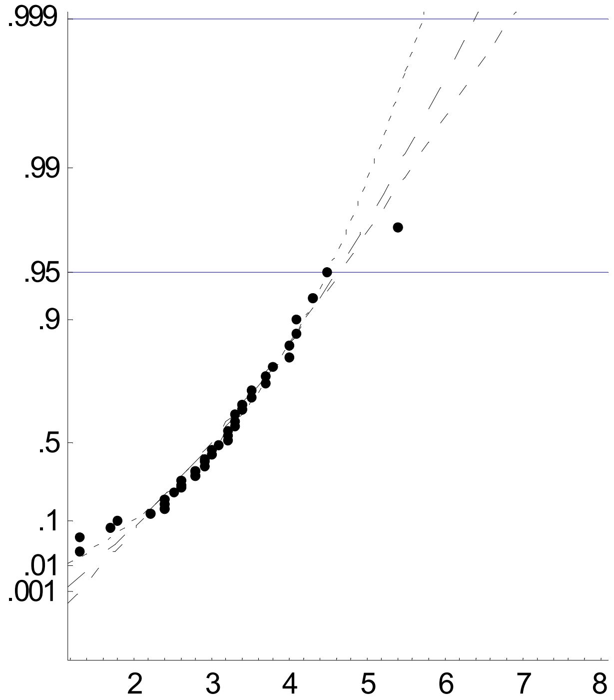
Distribution of the data points

```
datosord > SortAdataE;
                     i
probab > TableB ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢, 9i, 1, ndata=F;
                ndata + 1.
loglogprob > .LogA.LogAprobabEE;
```

```
points1 > 99datosordQ1U, probabQ1U==;
points2 > 99datosordQ1U, .LogA.LogAprobabQ1UEE==;
DoApoints1 > AppendToApoints1, 9datosordQiU, probabQiU=E, 9i, 2, ndata=E;
DoApoints2 > AppendToApoints2, 9datosordQiU, loglogprobQiU=E, 9i, 2, ndata=E;
p1 > ListPlotApoints1,
        Prolog Â 9PointSizeA.02E=,
        DisplayFunction Â Identity, PlotRange Â AllE;
pp3 > ListPlotApoints2,
        Prolog Â 9PointSizeA.02E=,
        DisplayFunction Â Identity,
        PlotRange Â AllE;
p3 > ShowApp3,
        Prolog Â 9PointSizeA.02E=,
        DisplayFunction Â IdentityI+,
        AxesOriginÂ90.,.1.5=,
        TicksÂ9Automatic,
                 99..834,".1"=,9.366,".5"=,92.25,".9"=,
                  92.97,".95"=,94.6,".99"===+M    E;
Draws in  normal and double-logaríthmic  scale
DoAfdistrib1AiE > PlotACDFAdistAiE, xE, 9x, xmin, xmax=,
            PlotStyle Â 9DashingA9.008 i^1.5 , .02 i=E=,
            DisplayFunction Â IdentityE,
        9i, 1, numdistributions=E;
DoAfdistrib2AiE > LogPlotACDFAdistAiE, xE, 9x, xmin, xmax=,
            PlotStyle Â 9DashingA9.008 i^1.5 , 0.02 i=E=,
            DisplayFunction Â IdentityE,
    9i, 1, numdistributions=E;
DoAfdistrib3AiE > PlotAEvaluateA.LogA.LogACDFAdistAiE, xEEEE, 9x, xmin, xmax=,
            PlotStyle Â 9DashingA9.008 i^1.5 , 0.02 i=E=,
            DisplayFunction Â IdentityE,
    9i, 1, numdistributions=E;
ShowA9p1, fdistrib1A1E, fdistrib1A2E, fdistrib1A3E=,
            AspectRatio Â 1.2,
              GridLines Â 9None, 90.95==,
            DisplayFunction Â $DisplayFunctionE;
```




```
ShowA9p3, fdistrib3A1E, fdistrib3A2E, fdistrib3A3E=,
        AspectRatio Â 1.2,
        AxesOrigin Â 9xmin, .3=,
        PlotRange Â 99xmin, xmax=, 9.3, 7==,
      GridLines Â 9None, 92.97, 6.9==,
        Ticks Â 9Automatic,
                    99.1.93, ".001"=, 9.1.527, ".01"=, 9..834, ".1"=, 9.366, ".5"=,
                    92.25, ".9"=, 92.97, ".95"=, 94.6, ".99"=, 96.9, ".999"=
                    ==,   I+ 99.21,".9999"=+M
          DisplayFunction Â $DisplayFunctionE;
```



Attachment 7 – MATHEMATICA notebook "FORM.nb"

FORM

This notebook compute the reliability index, failure probability and influence factors in level II, using the package "Reliability`Level2". In this package those variables are determined through the algorithm "Normal Tail Approxima tion" as is explained in the book of Madsen et al.: Methods of Structural Safety, pp. 94 and following.
The failure function of the limit state must be defined and, also, the independent basic variables given by a matrix with a row for each variable with the kind of distribution function assumed, the mean and the standard deviation.
The following names of distribution functions are implemented:

"normal" or "gauss",
"lognormal" or "lognor",
"gamma",
"logistic",
"uniform",
"weibull", and
"extremes" or "gumbel".

The iterations defined in the loop of the algorithm is stopped when the differences between two consecutive values i less than the error defined.

```
Needs["Reliability`Level2`"]
```

LImit State functio

Define the limit state function

```
gAz_E :>
 zAA1EE IzAA2EE + zAA3EEM + IzAA4EE . IIzAA2EE + zAA3EEM ® I2 + zAA5EE+ bMMM .
  zAA6EE IzAA7EE , zAA8EEM + L2 ® 8
```

Deterministic variable

```
b > 1;
L > 6.;
```

Ramdom variable

Define the variables matrix: distribution function, mean and standard deviation

```
m = {{"lognormal",    1,.1},
     {"normal",   0.00082   ,0.00082*0.05},
      {"lognormal",    560,560*0.054},
      {"normal",0.22,0.01}      ,
      {"lognormal",    30,30*0.167},
      {"lognormal",    1.2,.18},
      {"normal",   0.007,0.007*.1},
      {"gumbel",    .0008,.0008*0.6      }
     };
```

```
Failprob[m,g]
```

Obtain the reliability index beta ; the failure probabilities; and the sensitivity factor for each variable

```
beta > 3.80004, PhiI.betaM > 7.23367Å10.5 I 5 iterationsM

alpha > 9.0.442388, .0.223455, .0.230898, .0.216807, .0.0258585, 0.661535, 0.334996, 0.321747=
```

Attachment 8 – MATLAB package "Level2.m"

```
BeginPackage["Reliability`Level2`"]

Needs["Statistics`ContinuousDistributions`"]

Failprob::usage =
        "Work out the Hassofer-Lind  reliability index and the
        coefficients of sensitivity using the algorithm  Norman Tail
        Approximation given  in STRUCTURAL SAFETY  by
                Madsen et al., p94 . It admits a maximum of 10 variables  "

Characterization::usage =
        " Characterize the distribution functions given by a matrix "

Factores::usage =
        " Gives the sensitivity factors"

Begin["`Private`"]

Off[Part::partw]

Grad[s_, var_List] := D[s, #]& /@ var                (* Definition of Grad*)
invnormal[y_]:=Sqrt[2.] InverseErf[2y-1];

Failprob[matrix_,g_]:=
Module[
  {     n=First[Dimensions[matrix]],
        znew=muiter=Transpose[matrix][[2]],
        sigmaiter=Transpose[matrix][[3]],
           distnor=NormalDistribution[0.,1.],
           error= 10.^(-3),                (* error admited *)
           betanterior=0,
           ind=0,
           z,aux,gradiente,gradpart, ceta,muzeta,sigmazeta,alpha,beta,prob   },
  z=Array[a,n];
  gradiente=Grad[g[z],z] ;
  Characterization[matrix];   (* Characterization of the variables *)
  Do[                               (*  iterative cycle *)
        ziter=znew;
        partic=Take[{z[[1]]->ziter[[1]],z[[2]]->ziter[[2]],z[[3]]->ziter[[3]],
                z[[4]]->ziter[[4]],z[[5]]->ziter[[5]],z[[6]]->ziter[[6]],
                z[[7]]->ziter[[7]],z[[8]]->ziter[[8]],z[[9]]->ziter[[9]],
                z[[10]]->ziter[[10]]},n];
                                                              (*  iteration values *)
                Do[
                aux = invnormal[CDF[dist[i],ziter[[i]]]];
                sigmaiter[[i]] = PDF[distnor,aux]/PDF[dist[i], ziter[[i]] ];
                muiter[[i]] = ziter[[i]] - aux sigmaiter[[i]],
```

```
                        {i,n}];
                gradpart=gradiente/.partic;
                ceta=gradpart . ziter;
                muzeta=gradpart . muiter;
                sigmazeta=Sqrt[(gradpart^2).(sigmaiter^2)];
                alpha = -(gradpart*sigmaiter)/sigmazeta;
                beta = -(ceta-g[ziter]-muzeta)/sigmazeta;
                znew = muiter + beta(alpha * sigmaiter);
                prob=CDF[distnor,-beta];
                ind++
                If[Abs[beta-betanterior] < error,Break[],betanterior=beta],
     {10}];

 Print["beta = ",N[beta,5], ", Phi(-beta) = ",ScientificForm[prob]," ( \
",ind," iterations)" ];
 Print["alpha = ",N[alpha,3] ]
]

Factores:=
Module[
  {alpha,znew,muiter,sigmaiter},
  Print["alpha = ",N[alpha,4]];
  Print["znew = ",N[znew,4]];
  x=(znew-muiter)1./sigmaiter;
  Print["x = ",N[x,4]]
]

Characterization[matrix_]:=
Module[
  {w,par1,par2,a,
   n=First[Dimensions[matrix]]
  },
  Do[
    If[matrix[[i,1]]==="normal"||matrix[[i,1]]==="gauss",
                par1=matrix[[i,2]];
                par2=matrix[[i,3]];
        dist[i]=NormalDistribution[par1,par2],
    If[matrix[[i,1]]==="lognormal"||matrix[[i,1]]==="lognor",
                        w=(matrix[[i,2]]^2 + matrix[[i,3]]^2)/matrix[[i,2]]^2;
                par1=.5Log[matrix[[i,2]]^2/w];
                par2=Sqrt[Log[w]];
        dist[i]=LogNormalDistribution[par1,par2],
    If[matrix[[i,1]]==="gamma",
                        par1=matrix[[i,2]]^2/matrix[[i,3]]^2;
                par2=matrix[[i,3]]^2/matrix[[i,2]];
        dist[i]=GammaDistribution[par1,par2],
    If[matrix[[i,1]]==="logistic",
                        par1=matrix[[i,2]];
                par2=matrix[[i,3]] Sqrt[3.]/Pi;
        dist[i]=LogisticDistribution[par1,par2],
```

```
      If[matrix[[i,1]]==="uniform",
                              par1=matrix[[i,2]]-Sqrt[3.] matrix[[i,3]] ;
                    par2=matrix[[i,2]]+Sqrt[3.] matrix[[i,3]] ;
          dist[i]=UniformDistribution[par1,par2],
      If[matrix[[i,1]]==="weibull",
                              par1=a /. FindRoot[    matrix[[i,2]]^2/ matrix[[i,3]]^2==
                                                                                 Gamma[1+1/a]/(Gamma[1+2/a]-
Gamma[1+1/a]^2),{a,.1}];
                    par2=matrix[[i,2]]/Gamma[1+1/par1];
          dist[i]=WeibullDistribution[par1,par2],
     If[matrix[[i,1]] === "extremes" || matrix[[i,1]] ==="gumbel",
                    par1 = matrix[[i,2]] - N[(Sqrt[6]*EulerGamma*matrix[[i,3]])/Pi];
                              par2 = N[matrix[[i,3]]*Sqrt[6]/Pi];
                    dist[i] =  ExtremeValueDistribution[par1, par2],
     Print["Error: the distribution ", i ," (",matrix[[i,1]],
                  ") is not defined. "];
                    Return[]]]]]]]],
   {i,n}]

   ]

   End[]


EndPackage[]\.1a
```

Attachment 9 – MATHCAD sheet "FORM2.mcd"

MATHCAD sheet "FORM2.mcd", FORM for g(*X*)= *R* - *E* = 0 assuming general three parameter lognormal distribution LN(μ,σ,α) of *E* and *R*

1. Parameters for E and R: Highlighted regions $\mu E := 50$ $wE := 0.2$ $\alpha E := 1.14$ $\sigma E := wE \cdot \mu E$

Default distribution of *R* - two parameter lognormal $\mu R := 100$ $wR := 0.1$ $\alpha R := 3 \cdot wR + wR^3$ $\sigma R := wR \cdot \mu R$

2. Parameter C and skewness α:

Distribution parameter C given by the skewness α:

$$C(\alpha) := \frac{\sqrt[3]{\sqrt{\alpha^2+4}+\alpha} - \sqrt[3]{\sqrt{\alpha^2+4}-\alpha}}{\sqrt[3]{2}}$$

Distribution bound x0 (μ - 6 σ for zero α):

$$x0(\mu,\sigma,\alpha) := \begin{cases} \mu - \dfrac{\sigma}{C(\alpha)} & \text{if } \alpha \neq 0 \\ \mu - 6\sigma & \text{otherwise} \end{cases}$$

Check:

$C(0) = 0$

$x0(\mu R, \sigma R, \alpha R) = -8.527 \times 10^{-14}$

$x0(\mu E, \sigma E, \alpha E) = 22.522$

Transformation of parameters $m(\sigma,\alpha) := -\ln(|C(\alpha)|) + \ln(\sigma) - (0.5)\cdot\ln\left(1 + C(\alpha)^2\right)$ $\quad s(\alpha) := \sqrt{\ln\left(1 + C(\alpha)^2\right)}$

3. Probability density of E and R (for any α):

$$\phi E(x) := \begin{cases} \text{dlnorm}[\text{sign}(\alpha E)\cdot(x - x0(\mu E,\sigma E,\alpha E)), m(\sigma E,\alpha E), s(\alpha E)] & \text{if } \alpha E \neq 0 \\ \text{dnorm}(x,\mu E,\sigma E) & \text{if } \alpha E = 0 \end{cases}$$

$$\phi R(x) := \begin{cases} \text{dlnorm}[\text{sign}(\alpha R)\cdot(x - x0(\mu R,\sigma R,\alpha R)), m(\sigma R,\alpha R), s(\alpha R)] & \text{if } \alpha R \neq 0 \\ \text{dnorm}(x,\mu R,\sigma R) & \text{if } \alpha R = 0 \end{cases}$$

4. Distribution function of E and R (for any α) :

$$\Phi E(x) := \begin{cases} 0.5\cdot(1 - \text{sign}(\alpha E)) + \text{sign}(\alpha E)\,\text{plnorm}[\text{sign}(\alpha E)\cdot(x - x0(\mu E,\sigma E,\alpha E)), m(\sigma E,\alpha E), s(\alpha E)] & \text{if } \alpha E \neq 0 \\ \text{pnorm}(x,\mu E,\sigma E) & \text{if } \alpha E = 0 \end{cases}$$

$$\Phi R(x) := \begin{cases} 0.5\cdot(1 - \text{sign}(\alpha R)) + \text{sign}(\alpha R)\,\text{plnorm}[\text{sign}(\alpha R)\cdot(x - x0(\mu R,\sigma R,\alpha R)), m(\sigma R,\alpha R), s(\alpha R)] & \text{if } \alpha R \neq 0 \\ \text{pnorm}(x,\mu R,\sigma R) & \text{if } \alpha R = 0 \end{cases}$$

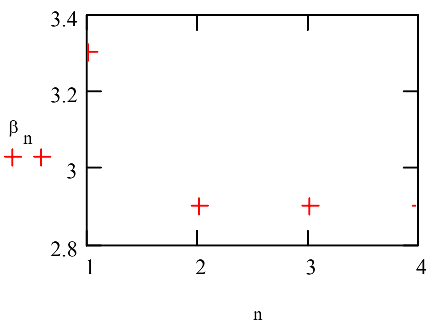
5. FORM iteration process: Guess values: $xE := \mu E + 0.005(\mu R - \mu E)$ $\quad xR := xE$ Number of iterations $n := 1..4$

$$\beta_n := \left| \begin{array}{l} xR \leftarrow xR \\ xE \leftarrow xE \\ \text{for } i \in 1..n \\ \quad \left| \begin{array}{l} \sigma Re \leftarrow \dfrac{\text{dnorm}(\text{qnorm}(\Phi R(xR),0,1),0,1)}{\phi R(xR)} \\ \mu Re \leftarrow xR - \sigma Re\cdot\text{qnorm}(\Phi R(xR),0,1) \\ \sigma Ee \leftarrow \dfrac{\text{dnorm}(\text{qnorm}(\Phi E(xE),0,1),0,1)}{\phi E(xE)} \\ \mu Ee \leftarrow xE - \sigma Ee\cdot\text{qnorm}(\Phi E(xE),0,1) \\ \beta \leftarrow \dfrac{\mu Re - \mu Ee}{\left(\sigma Re^2 + \sigma Ee^2\right)^{0.5}} \\ aR \leftarrow \dfrac{-\sigma Re}{\left(\sigma Re^2 + \sigma Ee^2\right)^{0.5}} \\ aE \leftarrow \dfrac{\sigma Ee}{\left(\sigma Re^2 + \sigma Ee^2\right)^{0.5}} \\ xR \leftarrow \mu Re + aR\cdot\beta\cdot\sigma Re \\ xE \leftarrow \mu Ee + aE\cdot\beta\cdot\sigma Ee \end{array} \right. \\ \beta \end{array} \right.$$

$\beta_n =$
3.304
2.904
2.904
2.904

$pf := \text{pnorm}(-\beta, 0, 1)$

$pf_n =$
$4.77\cdot10^{-4}$
$1.843\cdot10^{-3}$
$1.844\cdot10^{-3}$
$1.844\cdot10^{-3}$



Attachment 10 – MATHCAD sheet "FORM7.mcd"

MATHCAD sheet "FORM7" for calculation of the reliability index β and failure probability assuming a non-linear limit state function g(X) = a0 + a1*X1(a2*X2 + a3*X3) + a4*X4 (a5*X5+a6X6+a7X7) and general three parameter lognormal distribution LN(μ,σ,α) of basic variables *X*1, *X*2, *X*3, *X*4, *X*5, *X*6 and *X*7

A General three-parameter lognormal distribution for any α

1. Parameter C and skewness α:

Distribution parameter C given by the skewness α:
$$C(\alpha) := \frac{\sqrt[3]{\sqrt{\alpha^2+4}+\alpha} - \sqrt[3]{\sqrt{\alpha^2+4}-\alpha}}{\sqrt[3]{2}}$$

Check: $\boxed{C(0) = 0}$

Distribution bound x0 (μ - 6 σ for zero α):
$$x0(\mu,\sigma,\alpha) := \begin{cases} \mu - \dfrac{\sigma}{C(\alpha)} & \text{if } \alpha \ne 0 \\ \mu - 6\sigma & \text{otherwise} \end{cases}$$

$\boxed{x0(0,1,1) = -3.1038}$

2. Probability density ϕ and distribution function Φ (for any α):

Standardised variable: $u(x,\mu,\sigma) := \dfrac{x-\mu}{\sigma}$ Transformed standardised variable:

$$uu(x,\mu,\sigma,\alpha) := \begin{cases} \dfrac{\ln\left(\left|u(x,\mu,\sigma) + \dfrac{1}{C(\alpha)}\right|\right) + \ln\left(|C(\alpha)|\cdot\sqrt{1+C(\alpha)^2}\right)}{\mathrm{sign}(\alpha)\cdot\sqrt{\ln\left(1+C(\alpha)^2\right)}} & \text{if } \alpha \ne 0 \\ u(x,\mu,\sigma) & \text{otherwise} \end{cases}$$

$\boxed{uu(50,50,10,0) = 0}$

Density probability function:

$$\phi(x,\mu,\sigma,\alpha) := \begin{cases} \dfrac{\mathrm{dnorm}(uu(x,\mu,\sigma,\alpha),0,1)}{\sigma\cdot\left|u(x,\mu,\sigma) + \dfrac{1}{C(\alpha)}\right|\cdot\sqrt{\ln\left(1+C(\alpha)^2\right)}} & \text{if } \alpha \ne 0 \\ \dfrac{\mathrm{dnorm}(uu(x,\mu,\sigma,\alpha),0,1)}{\sigma} & \text{otherwise} \end{cases}$$

$\boxed{\phi(50,50,1,0) = 0.3989}$

Distribution function: $\Phi(x,\mu,\sigma,\alpha) := \mathrm{pnorm}(uu(x,\mu,\sigma,\alpha),0,1)$

$\boxed{\Phi(100,100,10,0) = 0.5}$

B FORM method for determination ofthe reliability index β and probability pf

Coefficients a0, a1, a2, a3, a4, a5, a6 and a7 of the limit state functions and

Input parameters for basic variables {X}={X1, X2, X3, X4, X5,X6 and X7}

Three parameter lognormal distribution LN(μ,σ,α) for any α, if α= 0 then the normal distribution is used . When X1 and X4 are model uncertainties then LN(μ,σ) is used.

$$\mathrm{Index} := \begin{pmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{pmatrix} \quad a := \begin{pmatrix}0.\\1\\1\\1\\-1\\1\\1\\1\end{pmatrix} \quad \mu := \begin{pmatrix}0\\1.\\100\\10\\1\\50\\10\\0.01\end{pmatrix} \quad \sigma := \begin{pmatrix}1\\0.05\\10\\1\\0.05\\10\\5\\0.01\end{pmatrix} \quad \alpha := \begin{pmatrix}0\\0.15\\0.301\\0.301\\0.15\\0.608\\1.14\\0\end{pmatrix}$$

$i := 1..7$

$x0_i := x0(\mu_i,\sigma_i,\alpha_i)$

$x0_1 = -8.3195\times 10^{-4}$

$x0_2 = -8.5265\times 10^{-14}$

Check of the bounds $x0_3 = -8.8818\times 10^{-15}$ $x0_4 = -8.3195\times 10^{-4}$ $x0_5 = 0$ $x0_6 = -3.7388$ $x0_7 = -0.05$

The check of the initial gues values $x_i := \mu_i$

$$x_1 := \frac{-\left[a_0 + a_4\cdot x_4\cdot\left(a_5\cdot x_5 + a_6\cdot x_6 + a_7\cdot x_7\right)\right]}{a_1\cdot\left(a_2\cdot x_2 + a_3\cdot x_3\right)}$$

$x_1 = 0.5455$

187

4. FORM iteration procedure:

Probability of failure pf is determined from the reliability index β

Number of iterations $n := 1..5$

The initial guess values of X

The value x2 is calculated from g (X)= 0

Equivalent normal distributions of the basic variables X.

Standardised variables

Derivatives of g(X):

Reliability index:

Sensitivity factors:

New design point to be used in the next iteration, go back to the section 5 and use this data in a new run

$$
\begin{aligned}
\beta_n := \; & x \leftarrow \mu \\
& x_1 \leftarrow \frac{-\left[a_0 + a_4 \cdot x_4 \cdot \left(a_5 \cdot x_5 + a_6 \cdot x_6 + a_7 x_7\right)\right]}{a_1 \cdot \left(a_2 \cdot x_2 + a_3 \cdot x_3\right)} \\
& \text{for } j \in 1..n \\
& \quad \text{for } i \in 1..7 \\
& \quad\quad \sigma e_i \leftarrow \frac{\mathrm{dnorm}\left(\mathrm{qnorm}\left(\Phi\left(x_i, \mu_i, \sigma_i, \alpha_i\right), 0, 1\right), 0, 1\right)}{\phi\left(x_i, \mu_i, \sigma_i, \alpha_i\right)} \\
& \quad\quad \mu e_i \leftarrow x_i - \sigma e_i \cdot \mathrm{qnorm}\left(\Phi\left(x_i, \mu_i, \sigma_i, \alpha_i\right), 0, 1\right) \\
& \quad\quad u_i \leftarrow \frac{x_i - \mu e_i}{\sigma e_i} \\
& \quad g_1 \leftarrow a_1 \cdot \left(a_2 \cdot x_2 + a_3 \cdot x_3\right) \sigma e_1 \\
& \quad g_2 \leftarrow a_1 \cdot a_2 \cdot x_1 \cdot \sigma e_2 \\
& \quad g_3 \leftarrow a_1 \cdot a_3 \cdot x_1 \cdot \sigma e_3 \\
& \quad g_4 \leftarrow a_4 \cdot \left(a_5 \cdot x_5 + a_6 \cdot x_6\right) \cdot \sigma e_4 \\
& \quad g_5 \leftarrow a_4 \cdot a_5 \cdot x_4 \cdot \sigma e_5 \\
& \quad g_6 \leftarrow a_4 \cdot a_6 \cdot x_4 \cdot \sigma e_6 \\
& \quad g_7 \leftarrow a_4 \cdot a_7 \cdot x_4 \cdot \sigma e_7 \\
& \quad \beta \leftarrow \frac{-(g \cdot u)}{\left(\overrightarrow{g \cdot g}\right)^{0.5}} \\
& \quad \text{for } i \in 1..7 \\
& \quad\quad aa_i \leftarrow \frac{g_i}{\left(\overrightarrow{g \cdot g}\right)^{0.5}} \\
& \quad\quad x_i \leftarrow \mu e_i - aa_i \cdot \beta \cdot \sigma e_i \\
& \quad x_1 \leftarrow \frac{-\left[a_0 + a_4 \cdot x_4 \cdot \left(a_5 \cdot x_5 + a_6 \cdot x_6 + a_7 x_7\right)\right]}{a_1 \cdot \left(a_2 \cdot x_2 + a_3 \cdot x_3\right)} \\
& \beta
\end{aligned}
$$

Iteration of the reliability index β

$$\beta = \begin{pmatrix} 0 \\ 2.9022 \\ 2.9303 \\ 2.9292 \\ 2.929 \\ 2.9289 \end{pmatrix}$$

$pf := \mathrm{pnorm}\left(-\beta_5, 0, 1\right)$

$pf = 1.7006 \times 10^{-3}$

Probability of failure pf

Attachment 11 – EXCEL sheet "FORM7.xls"

EXCEL sheet RORM7.xls - FORM iterative computation of the reliability index beta

Limit state function g(X)= a0+a1*X1*(a2*X2+a3*X3)+a4*X4*(a5*X5+a6*X6+a7*X7)

Basic variables Xi are approximated by general three-parameter lognormal distribution LN(mu,sigma, skew), which becomes automatically normal distribution when skew = 0.

Note that: skew = 3sigma/mu for two-parameter lognormal distribution
skew = 1.14 for Gumbel distribution
skew = 2sigma/mu for Gamma distribution

A	B	C	D	E	F	G	H	I	J	K	L
		Input data			The initial x1 is automatically calculated, do not change x1						
i	ai		Xi		Guess of x2		80,00		Lognormal of Xi		
0	0	mu	sigma	skew	C	x0	Design p.	u	uu	phi	PHI
1	1	1,000	0,050	0,150	0,050	0,00	0,963	-0,74	-0,73	6,3395	0,2319
2	1	100,000	10,000	0,301	0,100	0,00	65,451	-3,45	-4,20	0,0000	0,0000
3	1	0,010	0,010	0,301	0,100	-0,09	0,009	-0,07	-0,02	40,2671	0,4918
4	-1	1,000	0,050	0,150	0,050	0,00	1,035	0,71	0,72	5,9511	0,7642
5	1	30,000	10,000	0,000	0,000	infinity	54,089	2,41	2,41	0,0022	0,9920
6	1	5,000	2,000	1,140	0,364	-0,496	5,727	0,36	0,53	0,1581	0,7015
7	1	1,000	1,000	1,140	0,364	-1,748	1,061	0,06	0,24	0,3914	0,5943

Requirement for the design point, skewi>0 then x0i<xi, when skewi<0 then x0i>xi

Iteration of the FORM method - enter the new x manually instead of the initial x

	ai	xi	Equivalent norma		Deriv.		Sensitivity	New point		Partial factors
0	0	mue	sigmae	ui	gi	ui*gi	alpha	New x		x/mu
1	1	0,998	0,048	-0,73	3,15	-2,308	0,240	0,963		0,963
2	1	75,000	6,529	-1,46	6,286	-9,194	0,479	65,451	Enter the	0,655
3	1	0,010	0,010	-0,02	0,01	-2E-04	0,001	0,009	new xi	0,948
4	-1	0,998	0,052	0,72	-3,09	-2,228	-0,236	1,035	to x2	1,035
5	1	30,000	10,000	2,41	-10,4	-24,94	-0,788	54,089	to x7	1,803
6	1	4,567	2,195	0,53	-2,27	-1,201	-0,173	5,727	in H16	1,145
7	1	0,825	0,991	0,24	-1,03	-0,245	-0,078	1,061	to H21	1,061
				Sum	13,13	-40,12	1,000			

Number of iter.	1	beta in iter. n	3,0551	Macro procedure	
Required acc.	0,001	beta in iter. n-1	3,0551	1,8E-05	Accuracy reached

Probability of failure pf=PHI(-beta) - command =NORMDIST(-G34;0;1;1) 1,1E-03

The whole iteration procedure may be performed manually following instruction given in the cells K27 to K32 or using Macro "ITERATION".

Note that the initial gueas of variables X2 to X7 should be enteredto the cells H16 to H21.

Attachment 12 – MATLAB package "FORM7.m"

```
function pf = FORM7
% DESCRIPTION, 28.02.2005
%    FORM7a evaluates the probability of failure pf considering the limit state function
%
%              g(X)=A0 + A1*X1*(A2*X2+A3*X3) + A4*X4*(A5*X5+A6*X6+A7+A7)
%
% FUNCTIONS USED
%    LNDENS(x,ske,me,se), LNDIST(x,skr,mr,sr), NDENS(x), NDIST(x) and NDINV(p)
% INPUT
%    Input data (except A0) are loaded from the files A.dat (coefficients A), and X.dat
(parameters
%    of the basic variables X). All the basic variables are characterised by the mean m,
standard deviation s
%    and skewness sk (arbitrary). The FORM procedure approximates the basic variables by
general
%    lognormal distribution LN(m,s,sk), including normal distribution (for sk=0).
%
% OUPUT
%    val   : failure probability pf
% VERSION
%    MH, Czech Technical University in Prague, Klokner Institute, 28.2.2005
% Initialization
% %loading external data files
load A.dat, load X.dat, %The matrix X can be also defined in the comand window
A0=0;% additive constant (not included in the data file A.dot), default value A0=0. When A0
is different from 0,
   % numrical probles may arise. Then a new alternative initial point (for example modifying
resistance) may be choosen.
for i=1:1:7;
   x(i)=X(i,1); % Initial guess value of basic variables
end
x(1)=-(A0+A(4)*x(4)*(A(5)*x(5)+A(6)*x(6)+A(7)*x(7)))/(A(1)*(A(2)*x(2)+A(3)*x(3))); %
Initial guess value of x1
% FORM iterations
acc=0.001;delta=1;betap=0;j=0;%required accuracy (may be modified if required)initial
iteration parameters
while delta > acc %for j=1:1:5 %Iteration loop for a given accuracy (5 cycles are usually
sufficient for acc = 0.001)
   j=j+1; % The indicator of the number of cycles
   for i=1:1:7; % Loop for transformation of original distributions to equivalent normal
ditributions

se(i)=NDENS(norminv(LNDIST(x(i),X(i,1),X(i,2),X(i,3))))/LNDENS(x(i),X(i,1),X(i,2),X(i,3
));
      me(i)=x(i)-se(i)*norminv(LNDIST(x(i),X(i,1),X(i,2),X(i,3)));
   u(i)=(x(i)-me(i))/se(i); % Standardized variables
   end
      % Derivatives of g(X)
```

```
        g(1)=A(1)*(A(2)*x(2)+A(3)*x(3))*se(1);g(2)=A(1)*A(2)*x(1)*se(2);g(3)=
A(1)*A(3)*x(1)*se(3);
        g(4)=A(4)*(A(5)*x(5)+A(6)*x(6))*se(4); g(5)= A(4)*A(5)*x(4)*se(5);g(6)=
A(4)*A(6)*x(4)*se(6);
        g(7)= A(4)*A(7)*x(4)*se(7);
        % Auxiliary quantities
        gg=sqrt((g(1)*g(1)+g(2)*g(2)+g(3)*g(3)+g(4)*g(4)+g(5)*g(5)+g(6)*g(6)+g(7)*g(7)));
        gu=g(1)*u(1)+g(2)*u(2)+g(3)*u(3)+g(4)*u(4)+g(5)*u(5)+g(6)*u(6)+g(7)*u(7);
        % Reliability index
        beta =-gu/gg;
    for i=1:1:7; % Loop for determining sensitivity factors and a new design point
        aa(i)=g(i)./gg; % sensitivity factors
        x(i)=me(i)-beta.*aa(i).*se(i); % New design point
    end
        x(1)=-
(A0+A(4)*x(4)*(A(5)*x(5)+A(6)*x(6)+A(7)*x(7)))/(A(1)*(A(2)*x(2)+A(3)*x(3))); % Initial
guess value of x1
        delta=abs(beta-betap); % Difference of beats of two last cycles
        betap=beta; % Saving the current beta
end
%    Outputs
Number_of_iterations_snd_achieved_accuracy=[j,delta],
Alphas=[aa(1),aa(2),aa(3),aa(4),aa(5),aa(6),aa(7)],
Design_points=[x(1),x(2),x(3),x(4),x(5),x(6),x(7)],
Design_points_over_means=[x(1)/X(1,1),x(2)/X(2,1),x(3)/X(3,1),x(4)/X(4,1),x(5)/X(5,1),x(6
)/X(6,1),x(7)/X(7,1)],
beta, %a1=aa(1),a2=aa(2),a3=aa(3),a4=aa(4),a5=aa(5),a6=aa(6),a7=aa(7),% To be printed if
needed
pf=NDIST(-beta); % Answer of the function FORM7
% The end of the function FORM7
```

Attachment 13 – MATLAB function "Lndens (x,mu,sigma,sk)"

```
function val = Lndens (x,mu,sigma,sk)
%  DESCRIPTION
%     NDENS evaluates the one-dimensional normal density function.
%  CALL
%     val = Lndens (x,sk);
%     val = Lndens (x,sk,mu,sigma);
%  INPUT
%     x     : real vector of arguments
%     sk    : coefficient of skewness (must be given)
%     mu    : mean value; optional; default = 0.0 (i.e. standard)
%     sigma : std. dev. > 0; optional; default = 1.0 (i.e. standard)
%  OUPUT
%     val   : vector of normal density values for the x's
%  VERSION
%     Milan Holicky, Czech Technical University in Prague, Klokner Institute
%     18.09.1999
%  Initialization
if nargin < 3
  mu    = 0.0;
  sigma = 1.0;
end
%  Evaluate
x   = (x-mu)/sigma;                    % normalize
c=(0.5*sk+(sk^2/4+1)^0.5)^(1/3)-(-0.5*sk+(sk^2/4+1)^0.5)^(1/3);%constant of lognormal
if c==0; % sk=0
    x0=10^10;
  else
    x0=-1/c; %bound of the distribution
  end
if sk>0; %check of x range
  if x0>x
    error ('x out of range')
  else
  end
else
  if x0<x
    error('x out of range')
  else
  end
end
%
if abs(c)>0.0001; %if for c=0 (sk=0)
    tt=sign(sk)*(log(abs(x+1/c))+log(abs(c))+0.5*log(1+c^2))/((log(1+c^2))^0.5);
  else
    tt=x;
  end
if abs(c)>0.0001; %if for c=0 (sk=0)
```

```
  val = exp(-0.5*tt.^2)/(sqrt(2*pi)*sigma*abs(x+1/c)*(log(1+c^2))^0.5);
else
  val = exp(-0.5*tt.^2)/(sigma*sqrt(2*pi));
end
```

Attachment 14 – MATLAB function "Lndist (x,mu,sigma,sk) "

```
function val = Lndist (x,mu,sigma,sk)
% DESCRIPTION, 18.09.1999
%    LNDIST evaluates the one-dimensional lognormal distribution function.
% CALL
%    val = lndist (x,sk)
%    val = lndist (x,sk,mu,sigma)
% INPUT
%    x     : real vector of arguments.
%    sk    : coefficient of skewness; default = 0.0 (i.e. normal distribution)
%    mu    : mean value; optional; default = 0.0 (i.e. standard).
%    sigma : std. dev. > 0; optional; default = 1.0 (i.e. standard).
% OUPUT
%    val   : vector of the lognormal distribution evaluated at the x's.
% VERSION
%    Milan Holicky, Czech Technical University in Prague, Klokner Institute
% Initialization
if nargin < 3
  mu    = 0.0;
  sigma = 1.0;
end
x   = (x-mu)/sigma;            % standardize
c=(0.5*sk+(sk^2/4+1)^0.5)^(1/3)-(-0.5*sk+(sk^2/4+1)^0.5)^(1/3);%constant of lognormal
if c==0; % sk=0
    x0=10^10;
  else
    x0=-1/c; %bound of the distribution
  end
if sk>0; %check of x range
  if x0>x
    error ('x out of range')
  else
  end
else
  if x0<x
    error('x out of range')
  else
  end
end
if abs(c)>0.01; %if for c=0 (sk=0)
    tt=sign(sk)*(log(abs(x+1/c))+log(abs(c))+0.5*log(1+c^2))/((log(1+c^2))^0.5);
  else
    tt=x;
end
val = normcdf(tt);%(1+erf(tt/sqrt(2)))/2;    % transformed error function
```

Attachment 15 – MATLAB function "Ndens (x, mu,sigma)"

```
function val = ndens (x, mu,sigma)
%
%  DESCRIPTION
%     NDENS evaluates the one-dimensional normal density function.
%
%  CALL
%     val = ndens (x);
%     val = ndens (x, mu,sigma);
%
%  INPUT
%     x     : real vector of arguments
%     mu    : mean value; optional; default = 0.0 (i.e. standard)
%     sigma : std. dev. > 0; optional; default = 1.0 (i.e. standard)
%
%  OUPUT
%     val   : vector of normal density values for the x's
%
%  VERSION
%     Niels Jacob Tarp-Johansen
%     Department of Structural Engineering and Materials
%     Technical University of Denmark
%     15.06.1999
%


%  Initialization
if nargin < 2
   mu    = 0.0;
   sigma = 1.0;
end

%  Evaluate
x   = (x-mu)/sigma;                    % normalize
val = exp(-0.5*x.^2)/(sigma*sqrt(2*pi));
```

Attachment 16 – MATLAB function “Ndinv (p)”

```
function [x] = norminv(p,mu,sigma)
%NORMINV Inverse of the normal cumulative distribution function (cdf).
%   X = NORMINV(P,MU,SIGMA) returns the inverse cdf for the normal
%   distribution with mean MU and standard deviation SIGMA, evaluated at
%   the values in P.
%   Default values for MU and SIGMA are 0 and 1, respectively.
%
%   MH, Klokner Institute, CTU Prague 4.8.2003
%
if nargin < 2
   mu = 0;
end
if nargin < 3
   sigma = 1;
end
% Return NaN for out of range parameters or probabilities.
   sigma(sigma <= 0) = NaN;
   p(p < 0 | 1 < p) = NaN;
   x0 = -sqrt(2).*erfcinv(2*p);
   x = sigma.*x0 + mu;
```

Attachment 17 - MATLAB function "Ndist (x, mu,sigma)"

```
function val = ndist (x, mu,sigma)
%
% DESCRIPTION
%    NDIST evaluates the one-dimensional normal distribution function.
%
% CALL
%    val = ndist (x)
%    val = ndist (x, mu,sigma)
%
% INPUT
%    x     : real vector of arguments.
%    mu    : mean value; optional; default = 0.0 (i.e. standard).
%    sigma : std. dev. > 0; optional; default = 1.0 (i.e. standard).
%
% OUPUT
%    val   : vector of the normal distribution evaluated at the x's.
%
% VERSION
%    Niels Jacob Tarp-Johansen
%    Department of Structural Engineering and Materials
%    Technical University of Denmark
%    13.06.1999
% Initialization
if nargin < 2
  mu    = 0.0;
  sigma = 1.0;
end

% Evaluation
x   = (x-mu)/sigma;            % standardize
val = (1+erf(x/sqrt(2)))/2;    % transform error function
```

ANNEX C - CALIBRATION PROCEDURE

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Summary

The basic European standard for design of buildings and other engineering works, EN 1990 “Basis of structural design”, provides alternative design procedures, for which national choice is allowed. One of the most important questions concerns three fundamental combinations of actions for persistent and transient design situations in the Ultimate limit states. Simple example of generic structural member shows, that the alternative load combinations may lead to considerably different reliability levels. Probabilistic methods of structural reliability theory are used to identify characteristic features of each combination and to formulate general recommendations. It appears that further calibration studies concerning structures made of different materials are needed during the examination period of EN 1990 in order to analyse all possible consequences of national choice.

1 INTRODUCTION

1.1 Background materials

Each part of Eurocodes, including basic document EN 1990 [1], contains a number of the Nationally Determined Parameters (NDP) for which national choice is allowed. In accordance with the Guidance paper L concerning the Construction Products Directive an important two years period after date of availability of each Eurocode Part is allowed to fix the NDPs. However it is expected that calibration will continue during the coexistence period, which starts at the end of the National calibration period and lasts up to three years after the national publication of the last Part of a Package.

1.2 General principles

Basic concepts of code calibration are mentioned in Annex C of EN 1990 [1], in the International Standard ISO 2394 [2] and ISO 13822 [3]. Additional information may be found in the background document developed by JCSS [4] and in recently published handbook [5] to EN 1990 [1]. Guidance for application of probabilistic methods of structural reliability may be also found in working materials provided by JCSS [6] and in relevant literature listed in [5] and [6].

In general NDPs may be calibrated either by direct comparison or by probabilistic methods. Results of both approaches are usually combined with judgement (as mentioned in ISO 2394 [2]). In this study probabilistic approach is applied mainly, a direct comparison of load effects is shortly described in Appendix A to this contribution. Note that for the probabilistic calibration software products [7,8,9] can be used. In particular the programme [9] is intended for calibration purposes. Special purpose MATLAB functions and MATHCAD sheet attached to this Annex may be also used for calibration studies.

It should be noted that two aspects of calibration might be explicitly considered: reliability and economy (see also Handbook 1). However, the following text shall be primarily concerned with reliability of structures with respect to ultimate limit states. Additional calibration aspects may concern fire safety of structures (see Handbook 5) or other accidental design situations. In particular EN 1990 [1] requires that in the case of fire, the structural resistance shall be adequate for the required period of time.

To consider all the above-mentioned aspects of structural reliability, an appropriate design lifetime, design situations and limit states should be considered (as described in Handbook 1). Note that the basic lifetime for a common building is 50 years and that, in general, four design situations are identified: persistent, transient, accidental and seismic.

2 FUNDAMENTAL LOAD COMBINATIONS

In the following, the combination of three actions is considered: permanent action G, imposed load Q (leading) and wind W (accompanying). EN 1990 [1] for the fundamental combination of these loads in persistent and transient design situations introduces three alternative procedures denoted here A, B and C. The loads (actions) G, Q and W and their characteristic values G_k, Q_k and W_k denote generally load effects (for example internal bending moments) of appropriate loads (actions) and should be distinguished from the original loads (actions) themselves. However, when mutual proportions of loads (actions) and load effects are the same, then the distinction between load and load effects is not needed.

Design value of action effect E_d is obtained using the characteristic values G_k, Q_k and W_k and appropriate partial factors γ_G, γ_Q, γ_W and reduction factors ξ, ψ_Q and ψ_W as follows.

A. Considering the formula (6.10) in EN 1990 [1], the design value of action effect E_d is given as

$$E_d = \gamma_G\, G_k + \gamma_Q\, Q_k + \gamma_W\, \psi_W\, W_k \tag{1}$$

B. An alternative procedure is provided in EN 1990 [1] by twin expressions (6.10a) and (6.10b)

$$E_d = \gamma_G\, G_k + \gamma_Q\, \psi_Q\, Q_k + \gamma_W\, \psi_W\, W_k \tag{2}$$

$$E_d = \xi\, \gamma_G\, G_k + \gamma_Q\, Q_k + \gamma_W\, \psi_W\, W_k \tag{3}$$

The less favourable action effect from (2) and (3) should be considered. In equation (3) ξ is a reduction factor for unfavourable permanent actions G. Note that in equations (1) to (3) "+" generally implies "to be combined with".

C. In addition EN 1990 [1] allows further modification of alternative B, simplifying equation (2) by considering permanent loads only, thus the load effect is then

$$E_d = \gamma_G\, G_k \tag{4}$$

The less favourable action effect resulting from (3) and (4) is then considered. In addition to the combinations A, B, C provided in EN 1990 [1] (for recommended values γ_G = 1,35, γ_Q = 1,5) an additional combination may be also considered in the analysis to illustrate the sensitivity of the resulting reliability level to partial factors, and the possible effect of their reduction.

If the leading action is wind W, then in equations (1) and (2) instead of reducing wind action W by factor ψ_W, the imposed load Q should be reduced by the appropriate factor ψ_Q.

Factors γ_G, γ_Q and γ_W denote the partial factors of actions G, Q and W (the partial factors for both variable actions are equal, $\gamma_Q = \gamma_W$).

To investigate resulting load effects under various intensities of variable actions, the characteristic values of G_k, Q_k and W_k are related using quantities χ given as the ratio of variable actions Q_k+W_k to total load $G_k+Q_k+W_k$, and ratio k of accompanying action W_k to the main action Q_k

$$\chi = (Q_k+W_k)/(G_k+Q_k+W_k),\ k = W_k/Q_k \tag{5}$$

Note that a realistic range of χ is from 0,1 to 0,6. However in some cases the load ratio χ may be very low if not zero (e.g. underground garages).

For a given design value of the load effect E_d the characteristic values of individual actions G_k, Q_k, W_k can be expressed using variables χ and k as follows

$$G_k = \frac{E_d}{(\xi)\gamma_G + \dfrac{((\psi_Q)\gamma_Q + k(\psi_W)\gamma_W)\chi}{(1+k)(1-\chi)}},\ Q_k = \frac{\chi\, G_k}{(1+k)(1-\chi)},\ W_k = k\, Q_k \tag{6}$$

The factors ξ, γ_G and γ_Q indicated in the first relationship of (6) in brackets are applied in the same way (either yes or no) as in equations (1) to (4) for alternative combination rules A, B and C.

For alternative A, equation (1) is valid in the whole range $0 \le \chi \le 1$, whereas using alternative B, equation (2) is valid in the interval $0 \le \chi \le \chi_{lim,B}$ and equation (3) in the interval $\chi_{lim,B} \le \chi \le 1$. Correspondingly, for alternative C equation (4) is valid in the interval $0 \le \chi \le \chi_{lim,C}$ and equation (3) in the interval $\chi_{lim,C} \le \chi \le 1$. The limiting values $\chi_{lim,B}$ and $\chi_{lim,C}$ can be derived from equations (2) to (5) as follows

$$\chi_{lim,B} = \frac{\gamma_G(1-\xi)(1+k)}{\gamma_G(1-\xi)(1+k) + \gamma_Q(a-\psi_Q) + \gamma_W k(b-\psi_W)} \tag{7}$$

$$\chi_{lim,C} = \frac{\gamma_G(1-\xi)(1+k)}{\gamma_G(1-\xi)(1+k) + \gamma_Q a + \gamma_W kb} \tag{8}$$

where the auxiliary variable $a = 1$ and $b = \psi_W$ when for $k \le (1-\psi_Q)/(1-\psi_W)$ (imposed load Q is the leading action) and $a = \psi_Q$ and $b = 1$ when $k > (1-\psi_Q)/(1-\psi_W)$ (action W is the leading action).

EN 1990 allows through the National Annex, which will be published by national standardisation institution

- Which of the combination expression to use, and
- The specification of appropriate safety factors

Thus, the National Annexes should include the recommendation of one of the alternatives indicated in EN 1990 [1] for a fundamental combination of actions in the Ultimate limit states and partial factors γ_G and γ_Q for permanent and variable actions. Considering a generic structural member it will be shown that the choice of these nationally determined parameters may significantly affect the resulting reliability level. Partial and reduction factors γ, ψ and ξ recommended in EN 1990 [1] and used in this paper are summarized in Table 1.

Table 1. Partial and reduction factors.

Action	Partial factors γ	Combination factor ψ	Reduction factor ξ
Permanent G	1,35	1,0	0.85
Imposed Q	1,5	0,7	-
Climatic W	1,5	0,6	-

In addition to the factors indicated in Table 1 other values will be used to make comparison of Eurocode procedures with some national rules.

3 GENERIC STRUCTURAL MEMBER

In case of generic structural member it is assumed that the characteristic value R_k of the resistance R may be defined as the 5% fractile of R and the design value R_d as

$$R_d = R_k/\gamma_R \tag{9}$$

where γ_R denotes the global resistance factor (commonly expected to be within the range from 1 to 1,2). The significance of both values R_k and R_d is obvious from Figure 2, where the random variable R is described by the probability density function $\varphi_R(R)$, and the design value R_d is indicated as a particular value of R corresponding to a certain small probability p of being violated.

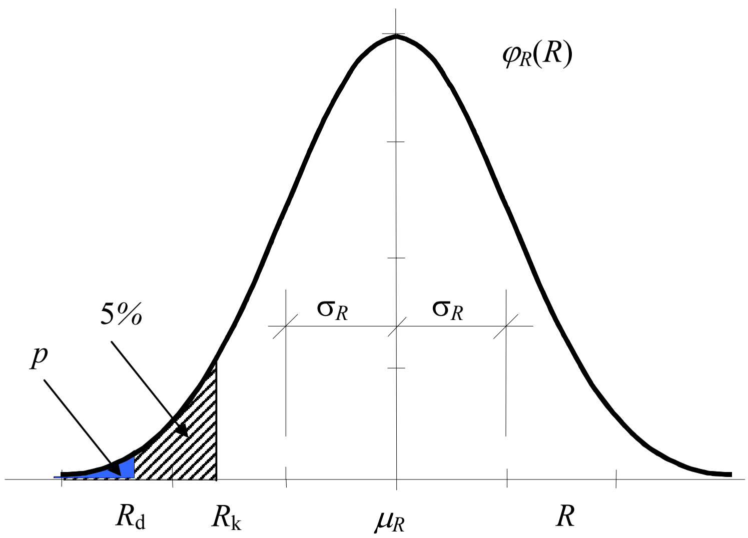

Figure 2. Random variable R, the characteristic value R_k and design value R_d.

In design calculation of a structural member the design value R_d of the resistance R is normally obtained by substituting design values X_{di} for the random variables X_i, thus

$$R_d = R(X_{di}) \tag{10}$$

This expression is also used in design of members for the generic structural member and for the different materials.

Table 2 shows the assumed values for the global resistance factor γ_R and the coefficient of variation V_R, used in the following reliability analysis for a generic cross-section.

Table 2 Global resistance factor γ_R, the coefficient of variation V_R and the mean factor ω.

	Middle value	Range
EN - global safety factor $\gamma_R = R_k / R_d$	1,15	1,0 – 1,3
BSI - global safety factor $\gamma_R = R_k / R_d$, BSI	1,10	1,0 – 1,20
The coefficient of variation V_R	0,15	0,10 – 0,25
The mean factor $\omega = \mu_R / R_k$	1,28	1,10 – 1,40

Note 1. The coefficient of variation V_R includes the variability of the model uncertainty assumed to have the coefficient of variability 0,05.
Note 2. The values are different for BSI codes reflecting the fact that lower values of μ_R are used for particular materials.

A middle values for the global safety factor γ_R = 1,15, for the coefficient of variation V_R = 0,15 and for the mean ratio $\omega = \mu_R/R_k$ = 1,28 are considered in the following example of a code condition.

4 PRINCIPLES OF RELIABILITY ANALYSIS

4.1 Limit state function

The most important step in reliability analysis is definition of a limit state function (reliability margin) $Z(\boldsymbol{X})$ separating safe and unsafe domain of basic variables $\boldsymbol{X}$. In this report the limit state function $Z(\boldsymbol{X})$ is considered as in a simple form as a difference between the resistance $R(\boldsymbol{X})$ and the load effect $E(\boldsymbol{X})$

$$Z(\boldsymbol{X}) = R(\boldsymbol{X}) - E(\boldsymbol{X}) = \theta_R R_0(\boldsymbol{X}) - \theta_E E_0(\boldsymbol{X}) \qquad (11)$$

where factor θ_R represents uncertainties of the resistance model $R_0(\boldsymbol{X})$ and factor θ_E represents uncertainties of the load effect model $E_0(\boldsymbol{X})$. The vector $\boldsymbol{X}$ denotes all the basic variables entering the expressions for the resistance $R(\boldsymbol{X})$ and the load effect $E(\boldsymbol{X})$. Taking into account general expressions (1) to (4) the load effect $E_0(\boldsymbol{X})$ may be written as

$$E(\boldsymbol{X}) = \theta_E (G_0 + Q_0 + W_0) \qquad (12)$$

Considering the limit state function given by equation (11) and expression (12) giving the load effect, it follows that basic variables R, G, Q, and W covering effects of model uncertainties are defined as follows

$$R = \theta_R R_0(\boldsymbol{X}),\ G = \theta_E G_0,\ Q = \theta_E Q_0,\ W = \theta_E W_0 \qquad (13)$$

Taking into account equation (13), the limit state function (11) may be written in a simple form as

$$Z(\boldsymbol{X}) = R - (G + Q + W) \qquad (14)$$

Note that the cumulative basic variables *R, G, Q, W* in equation (14) include effects of the factors θ_R and θ_E (see equation (13)).

4.2 Probabilistic models of basic variables

It is assumed that structural members are designed economically, which means that the design value of the resistance $R_d(\boldsymbol{X})$ equals the design value of the load effect $E_d(\boldsymbol{X})$

$$R_d(\boldsymbol{X}) = E_d(\boldsymbol{X}) \tag{15}$$

It should be noted that normally (due to several reasons) the design resistance $R_d(\boldsymbol{X})$ is greater than the design load effect $E_d(\boldsymbol{X})$, which may provide additional safety margin not considered here.

Assuming a certain set of partial and combination factors γ, ψ, and ξ, the design expression (15) can be used to specify the characteristic values X_k of each basic variable X. The probabilistic characteristics (the mean, standard deviation) of each basic variable X can be then related to its characteristic value X_k as indicated in Table 3.

Table 3. Probabilistic models of basic variables for time invariant reliability analysis using Turkstra's rule (combination of 50-year maximum of leading action and an annual maximum of accompanying action).

No.	Category of variables	Name of basic variables	Sym. X	Dim-ension	Distri-bution	Mean μ_X	St.dev. σ_X
1	Actions	Permanent	G_0	kN	N	G_k	$0,1\mu_X$
2		Imposed - 5 years	Q_0	kN/m^2	GU	$0,2Q_k$	$1,1\mu_X$
2		Imposed - 50 y.	Q_0	kN/m^2	GU	$0,6Q_k$	$0,35\mu_X$
3		Wind - 1 year	W_0	kN/m^2	GU	$0,3W_k$	$0,5\mu_X$
4		Wind - 50 year	W_0	kN/m^2	GU	$0,7W_k$	$0,35\mu_X$
5	Resistance	Resistance	R	kN/m^2	LN	$R_k + 1.65\sigma_R$	$0,15\mu_X$
6	Uncertainty	Uncertainty	θ_E	-	LN	1	0.05

Probabilistic models indicated in Table 3 are based on data available in the recommendation of JCSS [4,6] and literature [11,12,13,14]. As mentioned above the probabilistic characteristics indicated in Table 3 represent just conventional models that might be slightly conservative.

Note that the mean of a resistance R indicated in Table 3 in terms of the characteristic value R_k and the standard deviation σ_R may be assessed assuming a given coefficient of variation V_R using relationship

$$\mu_R = R_k \exp(1,65\ V_R) \tag{16}$$

Under this assumption the mean resistance factor ω considered in Table 2 is given as

$$\omega = \mu_R / R_k = \exp(1,65\ V_R) \tag{17}$$

Considering the coefficient of variation $V_R = 0.15$, the mean resistance factor becomes $\omega = 1.28$ as indicated in Table 2.

It should be emphasised that the probabilistic models of basic variables indicated in Table 3 are primarily intended as "conventional models" in time invariant reliability analysis of structural members using Turkstra's combination rule [10] (explained also in [2]) for the probabilistic calibration of the rules for combination of actions.

Conventional models indicated in Table 3 should enable the objective comparison of results of various reliability studies expected in the near future in connection with implementation of the present suite of Eurocodes into the national systems of design codes. However, when the reliability of different types of structural members under particular conditions is assessed, the proposed models in Table 3 may have to be adjusted to the concrete conditions of the analysed structural member.

4.3 Reliability measures

The probability of failure P_{f} is the basic reliability measure used in this study. It can be expressed on the basis of a limit state (performance) function $Z(X)$ defined in such a way that a structure is considered to survive if $Z(X) > 0$ and to fail if $Z(X) \leq 0$. An example of the function $Z(X)$ is given by equation (14). In a general case the failure probability P_{f} can be determined using the integral

$$P_{\mathrm{f}} = \mathrm{Prob}(Z \leq 0) = \int_{g(X)\leq 0} \varphi_g(X)\mathrm{d}X \tag{18}$$

where $\varphi_g(X)$ denotes joint probability density distribution of the basic variable X, which may not be , however, available.

Assume that both the resistance $R(X)$ and the load effect $E(X)$ represent a single variable X used to analyse structural performance (e.g. axial force or bending moment that is represented by $R(X)$ and $E(X)$). Then the integration indicated in expression (18) may be simplified and the probability P_{f} can then be expressed as:

$$P_{\mathrm{f}} = \mathrm{Prob}(Z(x) \leq 0) = \int_{-\infty}^{\infty} \varphi_E(x)\Phi_R(x)\mathrm{d}x \tag{19}$$

where $\varphi_E(x)$ denotes the probability density function of $E(X)$, $\Phi_R(x)$ the distribution of $R(X)$. To use equation (19) both the probability density function $\varphi_E(Z)$ and the distribution function $\Phi_R(x)$ must be known (at least in an approximate form). Simplified procedure based on expression (19) is used in this study.

Note that there are commercially available software products (e.g. VaP, COMREL), which can be used to determine the failure probability P_{f} in more complicated cases than considered here (when expression (19) cannot be used). These software products were used in this study to check results obtained by numerical integration based on expression (19).

In Annex C of EN 1990 an alternative measure of reliability is conventionally defined by the reliability index β, which is related to P_{f} as

$$P_{\mathrm{f}} = \Phi(-\beta) \tag{20}$$

where Φ is the cumulative distribution function of the standardised normal distribution. The relation between P_{f} and β is indicated in Table 4.

Table 4. Relation between β and P_{f}.

P_{f}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
β	1,28	2,32	3,09	3,72	4,27	4,75	5,20

Table C2 of EN 1990 recommends for the ultimate limit state of buildings over a fifty year design working life a target value of reliability index β_t = 3,8. If one year period is considered in reliability verification, then β_t = 4,7. Both the equivalent reliability measures, the failure probability P_f and the reliability index β, are used in this study.

4.4 Sensitivity factors

Sensitivity factors of the First Order Reliability Methods (FORM) are normally used [1,2] to calibrate design values of basic variables and partial safety factors. Considering the limit state function $Z(\boldsymbol{X})$ (reliability margin) given by equation (11), the sensitivity factors for the four cumulative variables R, G, Q, W can be defined in terms of their standard deviations σ_R, σ_G, σ_Q and σ_W as follows

$$\alpha_R = \frac{\sigma_R}{\sigma_g}, \ \alpha_G = \frac{\sigma_G}{\sigma_g}, \ \alpha_Q = \frac{\sigma_Q}{\sigma_g}, \ \alpha_W = \frac{\sigma_W}{\sigma_g} \tag{21}$$

where σ_g denotes the standard deviation of $Z(\boldsymbol{X})$ given as

$$\sigma_g = \sqrt{\sigma_R^2 + \sigma_G^2 + \sigma_Q^2 + \sigma_W^2} \tag{22}$$

In the following investigation the sensitivity factors α_R, α_G, α_Q and α_W defined by equation (21) are considered together with the failure probability P_f and the reliability index β. It should be underlined that α_R, α_G, α_Q and α_W defined by (21) refer to cumulative variables R, G, Q, W, which include effects of the factors of model uncertainties θ_R and θ_E (see equation (13)).

5 RESULTS FOR THE GENERIC CROSS-SECTION

5.1 One variable action

Results of the reliability analyses are presented in graphical form that indicates variation of the reliability index β, failure probability P_f, and sensitivity factors α_R, α_E, α_G, α_Q and α_W with the load ratio χ. In particular Figure 3 shows results of a simple case of one variable action only (the main variable action Q); Figure 3 indicates the variation of

- the reliability index β,
- failure probability P_f, and
- for expression 6.10 of EN 1990 sensitivity factors α_R, α_E, and partial sensitivity factors α_G, α_Q and α_W

with the load ratio χ.

For the analysis it has been assumed that a single variable action, the imposed load Q having the characteristic given in Table 3 is acting on the generic element only (i.e. k = 0.0). A middle value for the global safety factor γ_R = 1,15 and for coefficient of variation V_R = 0,15 have been considered.

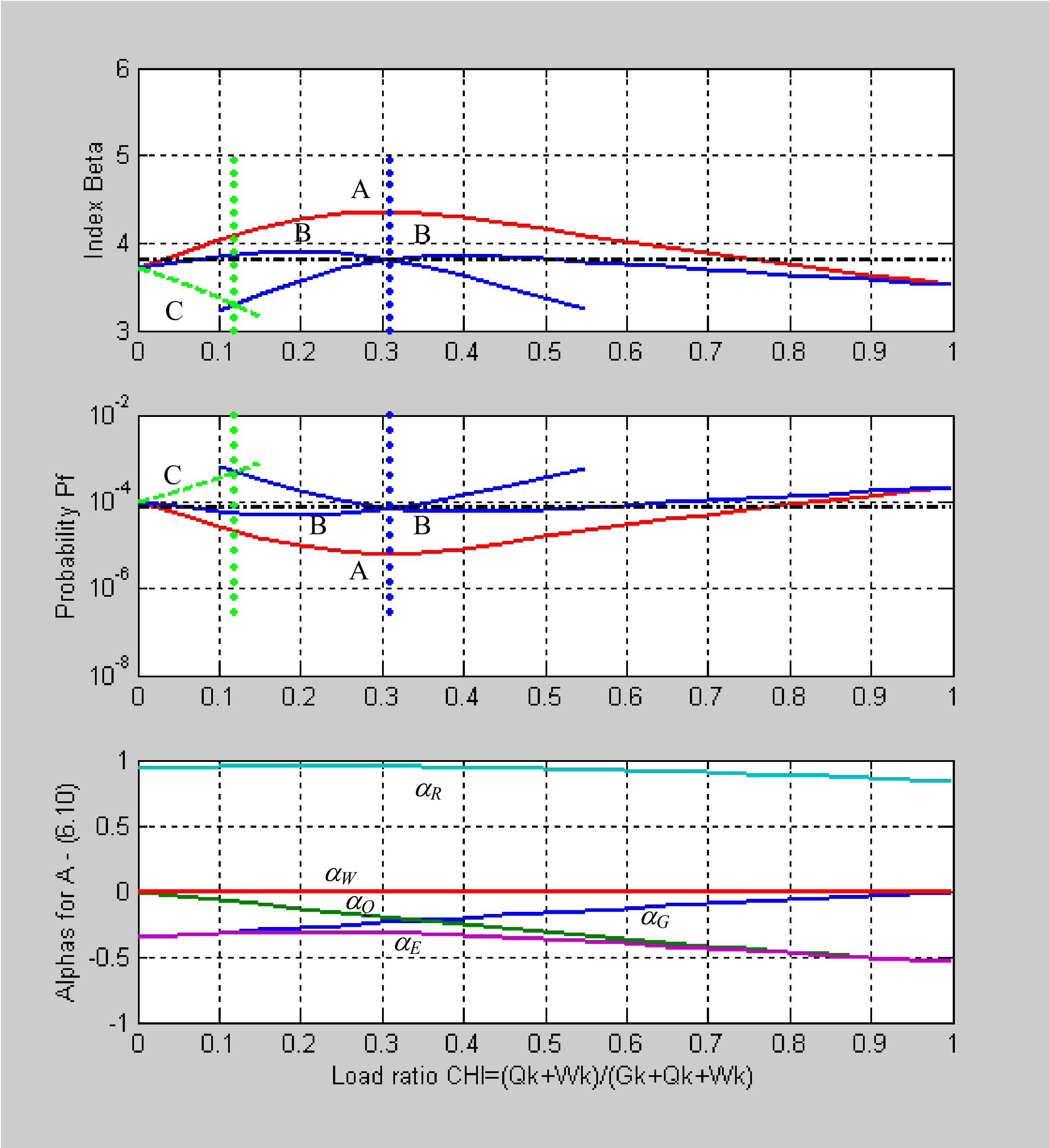

Figure 3. Variation of the reliability index β, the failure probability P_f and the sensitivity factors α_R, α_E, α_G, α_Q and α_W with the load ration χ for $k = 0$, for a generic cross-section assuming $\gamma_R = 1{,}15$ and the coefficient of variation $V_R = 0{,}15$.

It follows from Figure 3 that for the assumed higher coefficient of variation $V_R = 0{,}15$ only the combination rule A (i.e. expression (6.10) of EN 1990) [1] seems to be fully acceptable ($\beta > 3.8$ and $P_{f} < 7{,}23\times10^{-5}$) in the interval $0 < \chi < 0.8$, however the reliability level considerably varies with χ. In some cases the alternative A might lead to an uneconomic design.

The alternative B (i.e. expression (6.10a) and (6.10b) of EN 1990) is acceptable in a slightly shorter range of χ, $0 < \chi < 0.7$ than the variant A but provides obviously much more uniform distribution of reliability level with χ. Obviously it would lead to a more economic design than the alternative A. Alternative C (i.e. modified expression (6.10a) and (6.10b) of EN 1990 [1]) is providing rather low reliability level particularly for the interval $0 < \chi < 0.3$ and should not be used unless partial factors γ are changed.

Similar results were obtained in previous studies [11,12,13,14] of structural elements made of different materials (concrete and steel elements). These studies differ from the presented results primarily by the value of the partial factor γ_R and the coefficient of variation V_R (and also by the asymmetry of the distribution of R). Just the conclusions formulated above seem to be supported by a number of different material oriented examples.

Note that the sensitivity factor α_R increases to about $\alpha_R \sim 0,9$ while the factor α_E decreases, $\alpha_E > - 0,5$, indicating that the resistance gives a greater contribution to safety than intended by EN 1990. However this conclusion is strongly dependent on assumed coefficient of variation V_R. With increasing V_R the sensitivity factor α_R increases. It is interesting to note that than the sensitivity factors are very close to the values recommended in EN 1990 [1], i.e. $\alpha_R \sim 0,9$ $\alpha_E \sim -0,7$.

5.2 Two variable actions

A more general case when two variable actions (a leading imposed load Q, together with an accompanying action W) are acting is shown in Figure 4, which (similarly as Figure 3) shows the variation of the

- reliability index β,
- failure probability P_{f}, and
- for expression 6.10 of EN 1990 sensitivity factors α_R, α_E, α_G, α_Q and α_W

with the load ratio χ for k = 0,75 and the coefficient of variation V_R = 0,15.

The case considered for Figures 3 (i.e. k = 0, with a single imposed load Q acting) is extended so that a more detailed insight of the effect for the reliability parameters considered can be obtained. However Previous investigations [11,12] clearly show that reliability in case of two variable actions is considerably greater than reliability in case of one variable action.

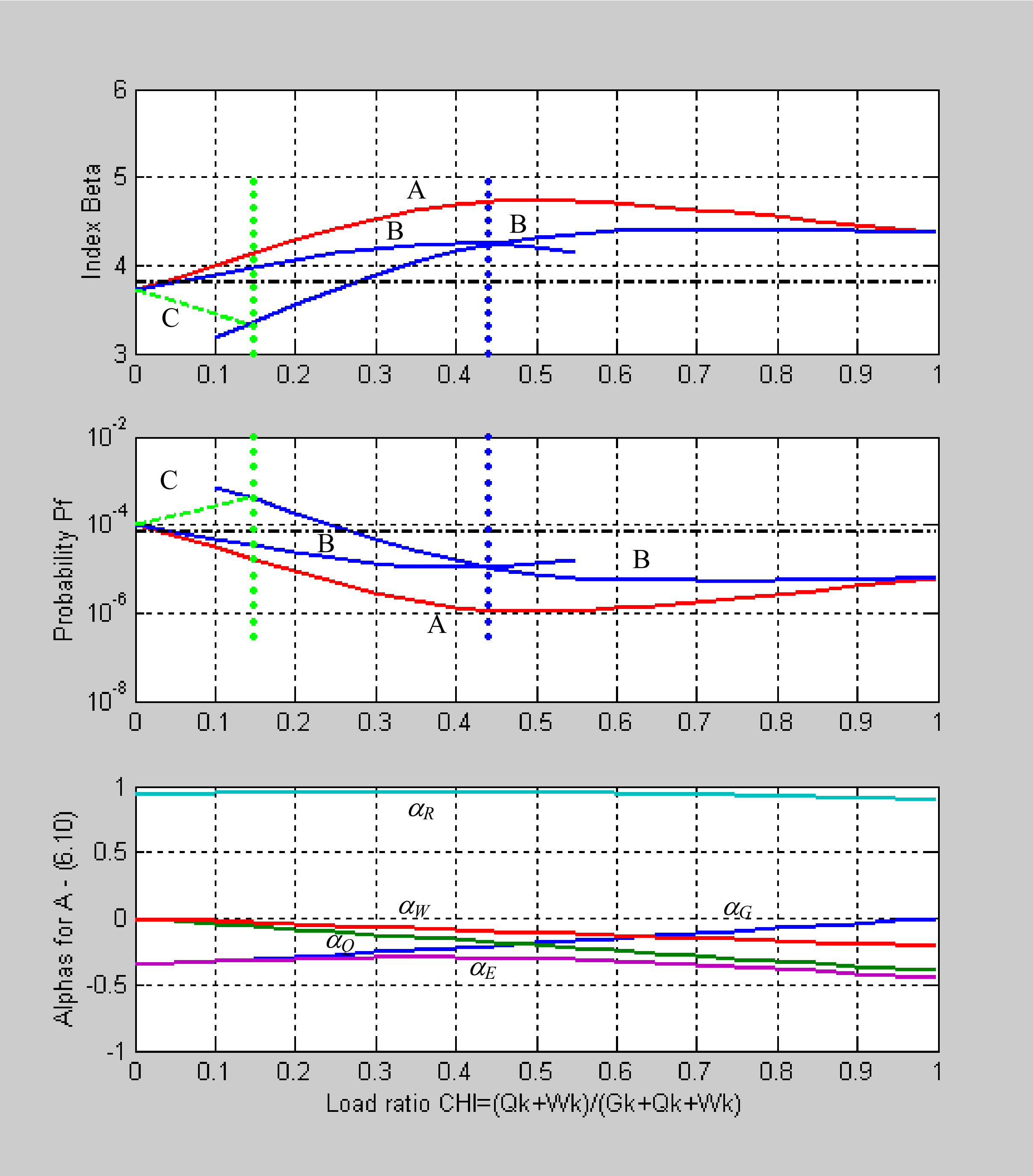

Figure 4. Variation of the reliability index β, the failure probability P_f and the sensitivity factors α_R, α_E, α_G, α_Q and α_W with the load ratio χ for k = 0.75, for a generic cross section assuming γ_R = 1,15 and the coefficient of variation V_R = 0,15.

It follows from Figure 4 that for the assumed coefficient of variation V_R = 0,15 and the consideration of two variable actions the reliability of the generic cross-section exposed to two variable actions is considerably greater than the reliability of the same cross-section exposed to one variable action only. This finding also indicates that the factor ψ_W may be rather high. Note that the sensitivity factors α_R seems to be slightly greater than the values α_R = 0,8 considered in EN 1990 [1] and α_E in absolute value is less than α_E = - 0,7 recommended in [1]. This finding depends on assumed variability of basic variables.

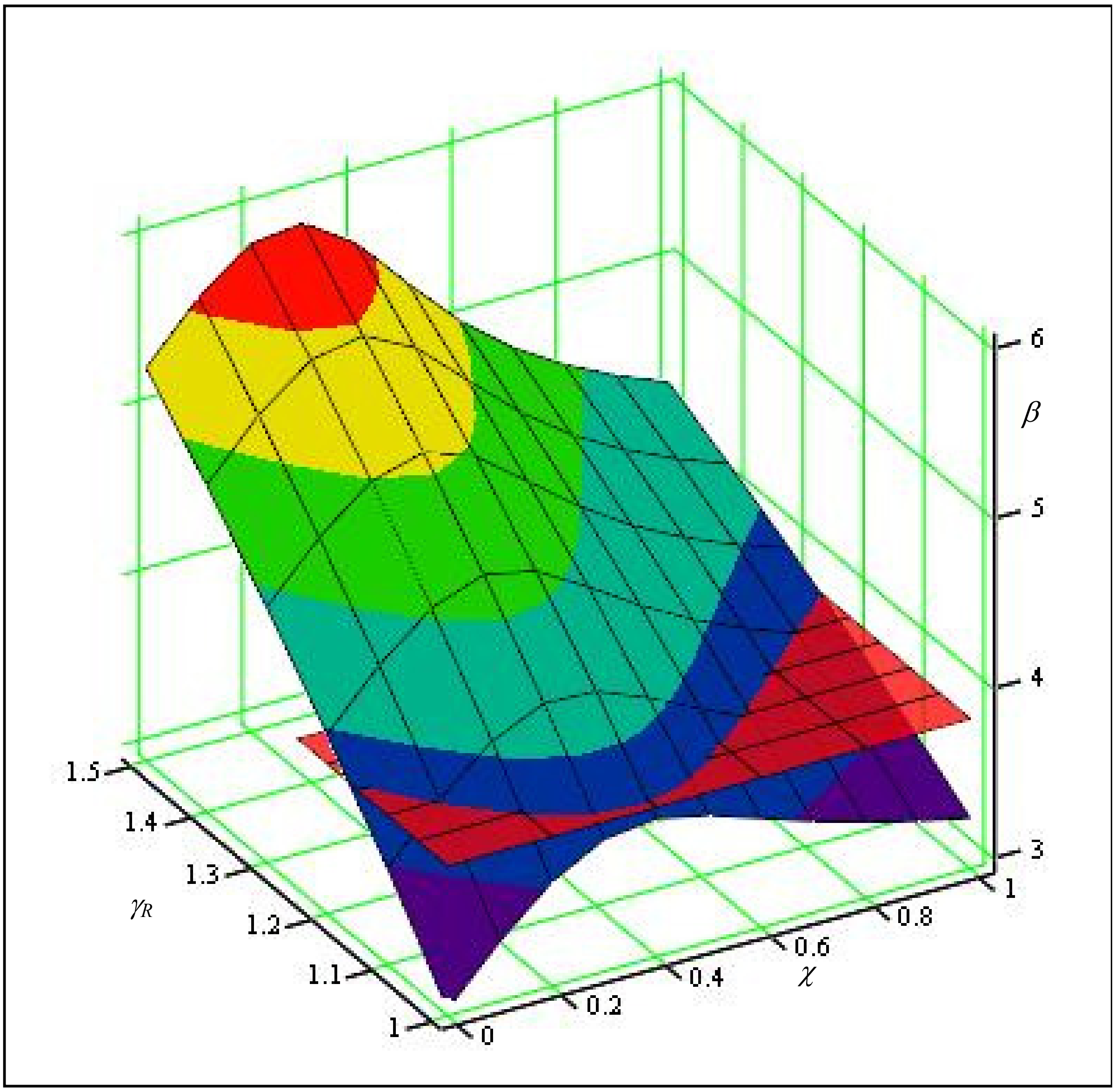

Figure 5. Variation of the reliability index β with the load ratio χ and partial factor for resistance γ_R and k = 0 (i.e. imposed load Q is the only variable action), for the generic-cross section assuming partial safety factors ψ_G = 1,35 and ψ_Q = 1,5, and the coefficient of variation V_R = 0,15.

It follows from Figure 5 that for the assumed variables the acceptable domain of the load ratio χ and the coefficient of variation V_R is limited by the contour line determined as an intersection of the β surface and the plain β = 3,8 in Figure 5. Obviously with increasing γ_R reliability index β increases, γ_R = 1.15 would be satisfactory for most of the practical range of the load ratio χ (for the load ratio χ.< 0,8).

6 CONCLUDING REMARKS

The newly available EN 1990 provides alternative design procedures and parameters that should be unambiguously specified in the National Annexes of Member States of CEN. These alternative design procedures lead in some cases to significantly different reliability levels. Preparation of National Annexes is therefore a complicated task for each Member State. Furthermore, the Eurocode standards recognise the responsibility of the regulatory authorities in each Member State and safeguard their right to determine values related to regulatory safety matters at national level.

Simple examples of a generic structural member confirm the results of the earlier studies that the reliability of structures, designed according to the alternative combination rules provided in EN 1990 by expressions (6.10), (6.10a) and (6.10b), may vary considerably. Expression (6.10) leads to the most reliable but in some cases uneconomical structures. Twin expressions (6.10a) and (6.10b) provide a lower but comparatively most uniform reliability level for all load ratios. Moreover, they seem to fully comply with EN recommendations (reliability index 3,8 for a 50-year time period). The lowest reliability is obtained from the third alternative, given by modified expression (6.10a) and expression (6.10b). This alternative seems to lead to a rather low reliability level, particularly for structures exposed mainly to a permanent load.

In order to make an unambiguous recommendation for National Annexes to EN 1990, further investigations are urgently needed. Obviously more complicated structural elements, made of various materials, should be analysed and compared. Such a calibration activity should preferably be organised on an international level. The short-term objective of these activities should be to develop the necessary background materials for preparation of the National Annexes. The long-term objective should be to further harmonization of the alternative design procedures considered during the next revision of the present generation of Eurocodes.

It is expected that further calibration studies concerning structures made of different materials will be needed during the examination period of EN 1990 (next few years) in order to analyse all possible consequences of national choice.

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APPENDIX A. DIRECT COMPARISON OF LOAD EFFECTS

Deterministic approach – the global load factor

Concept of the global load factor is sometimes used to compare various alternatives for load combination with no regard to a resistance of a structure. The deterministic global load factor γ_E follows directly from codified combination rules and given partial factors without any probabilistic consideration; it is simply expressed as

$$\gamma_E = E_\mathrm{d} / (G_\mathrm{k} + Q_\mathrm{k} + W_\mathrm{k}) \tag{A.1}$$

where the design load effect E_d is given by one of equations (1) to (4) depending on the combination rule considered (for example combination rules A, B or C). It follows from equation (1) to (4) and (A.1) that in general deterministic γ_E may be expressed as

$$\gamma_E = (1 - \chi)\ \gamma_G\,(\xi) + (\gamma_Q\,(\psi_Q) + k\,\gamma_W\,(\psi_W))\ \chi / (1 + k) \tag{A.2}$$

where the factors in brackets (ξ), (ψ_Q) and (ψ_W) are applied in accordance with the principles of appropriate combination rule. For example assuming that Q is the leading variable load and W is accompanying load, the combination rule A based on expression (6.10) of EN 1990 [1] the global factor γ_E follows from (1) and (A.2) as

$$\gamma_E = (1 - \chi)\ \gamma_G + (\gamma_Q + k\,\gamma_W\ \psi_W)\ \chi / (1 + k) \tag{A.3}$$

Similarly the global factors γ_E of other combination rules B and C may be obtained from general expression (A.2). It follows from (2) that equation (A.2) becomes

$$\gamma_E = (1 - \chi)\ \gamma_G + (\psi_Q\,\gamma_Q + k\,\gamma_W\ \psi_W)\ \chi / (1 + k) \tag{A.4}$$

When equation (3) is applied, then equation (A.2) becomes

$$\gamma_E = (1 - \chi)\ \gamma_G\,\xi + (\gamma_Q + k\,\gamma_W\ \psi_W)\ \chi / (1 + k) \tag{A.5}$$

When equation (4) is applied, then equation (A.2) becomes

$$\gamma_E = (1 - \chi)\ \gamma_G \tag{A.6}$$

Thus combination rule A is described by equation (A.3), combination B by equations (A.4) and (A.5), combination rule C by equations (A.4) and (A.6).

Figure A.1 shows the global factor γ_E for all three-combination rules A, B and C assuming the load factor $k = 0$ (two loads G and Q are considered only). It is interesting to note that the global load factor γ_E is strongly dependent on the load factor k. Figure A.2 shows the case of three variable actions G, Q and W assuming $k = 0{,}75$. Similar results may be obtained for any load ratio k. However, it is well recognised that decisive requirements (compare Figures 3 and 4) for calibration of reliability elements follow from combinations of two actions only (G and Q). Figure A.2 just illustrates variation of the global factor with the load ratio χ in the case of two variable actions.

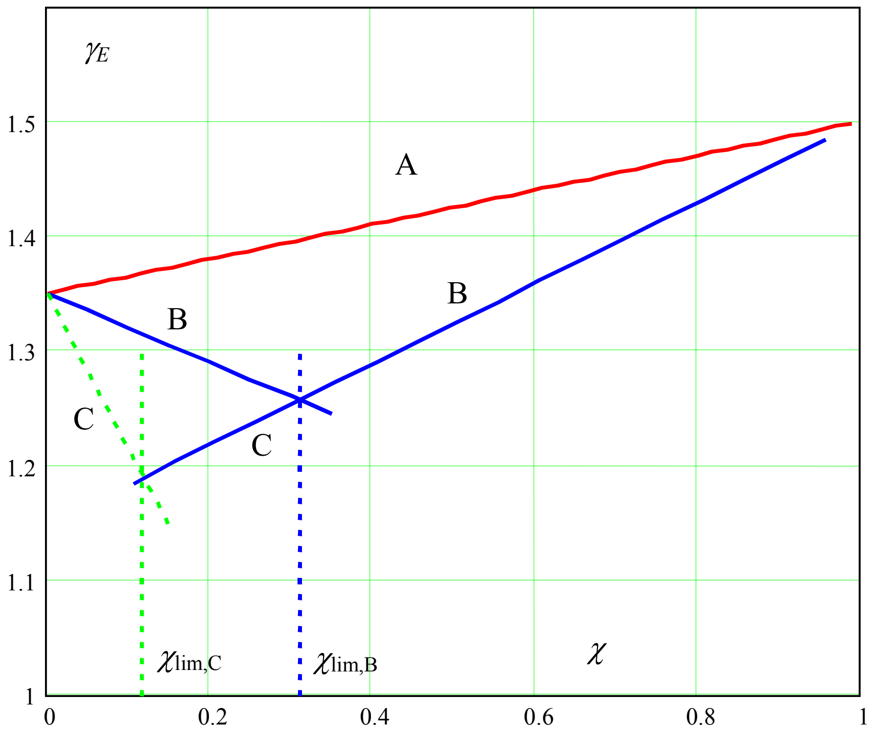


Figure A.1. The global load factor γ_E for the combination rules A, B and C assuming $k = 0$.

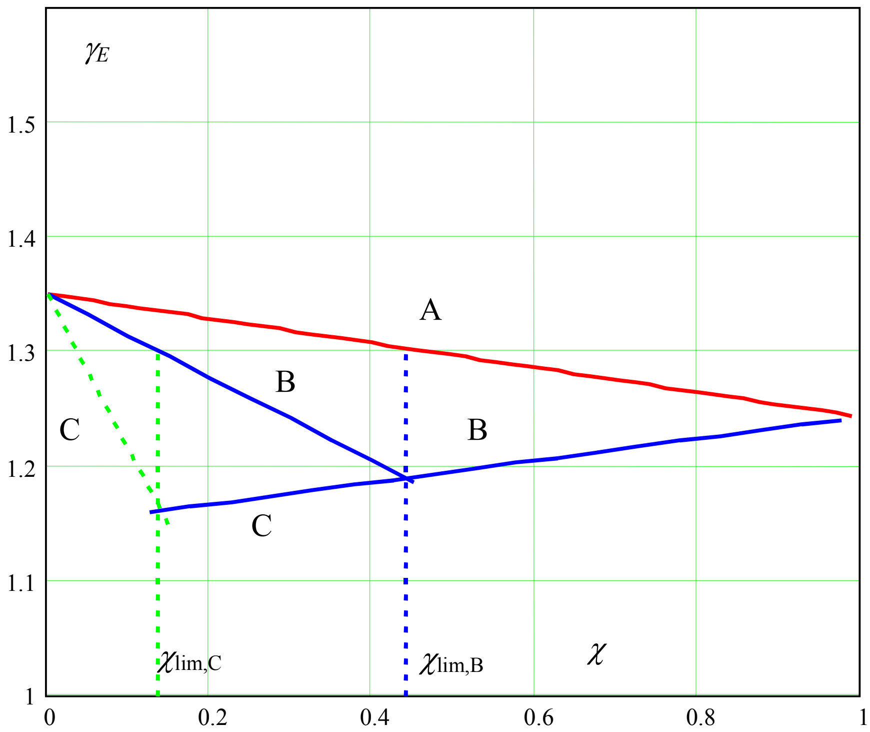


Figure A.2. The global load factor γ_E for the combination rules A, B and C assuming $k = 0{,}75$.

Probabilistic approach – the theoretical global load factor

Probabilistic approach to comparison of load effects E considers the probability p of E exceeding $E_d = \gamma_E\,(G_k + Q_k + W_k)$, thus the probability

$$p = \mathrm{P}(E > E_d) \qquad \text{(A.7)}$$

The probability p is obviously dependent on the global load factor γ_E determining the load effect E_d as follows from equation (A.1). When probabilistic models of actions G, Q and W are known, then for a given value of γ_E the probability p may be determined. Let us remind that in accordance with the principles of EN 1990 [1] (considering α_E = -0.7 and β = 3.8) the recommended value of the probability p given by equation (A.4) is

$$p_E = \Phi(\alpha_E\,\beta) = \Phi(-\,0{,}7\;3{,}8) = 0{,}004 \qquad \text{(A.8)}$$

Assuming probabilistic models of actions G, Q and W considered above in accordance with Table 3, variation of the theoretical load factors γ_E with the load factor χ for selected probabilities p is shown in Figure A.3 together with the deterministic load factors γ_E described above. In Figure A.3 full lines indicate the theoretical (probabilistic) load factors γ_E, the dashed lines indicate the deterministic load factors γ_E.

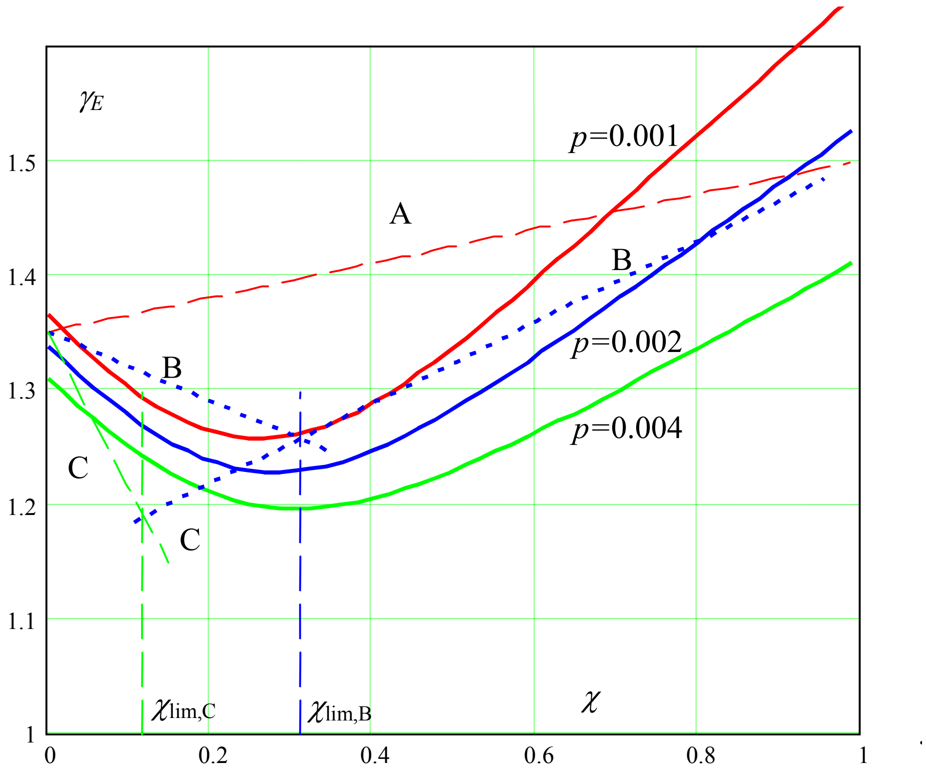


Figure A.3. The global load factor γ_E for the combination rules A, B and C assuming $k = 0$ and theoretical values of γ_E corresponding to selected probabilities of E exceeding E_d

Figure A.3 clearly indicates differences between the theoretical global factors γ_E determined using probabilistic approach and corresponding deterministic values described by equation (A.2). It follows from Figure A.3 that the deterministic values are greater than the theoretical values of γ_E corresponding to the probability $p = 0{,}004$ indicated in equation (A.5). Thus, the Eurocode combination rules seem to be on a safe side, in particular the combination rules A and B. Note that for small load ratios χ the combination rule C provides lower values of the global factor than the theoretical γ_E and, therefore, seems to be unsatisfactory.

Figure A.3 further indicates that the theoretical γ_E is better followed up by γ_E corresponding to the combination rule B than those corresponding to the combination rules A or C. In that sense direct comparison of load effects confirms conclusions of previous studies when both the load effect and resistance are taken into account.

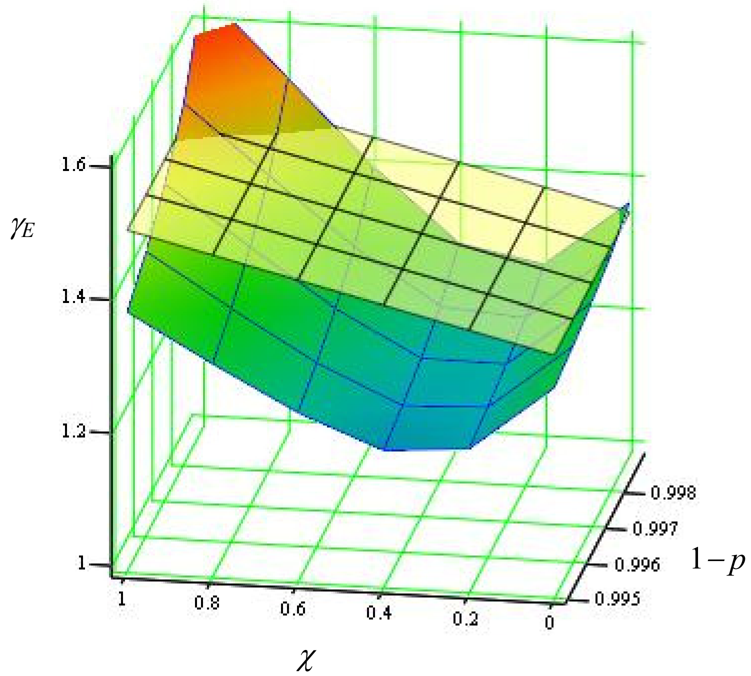


Figure A.4. Variation of the theoretical and deterministic global factor γ_E with the load ratio χ and the probability p assuming the combination rule A.

Variation of the global factor γ_E with the load ratio χ and the probability p clearly indicates that the combination rule A, represented in Figure A.4 by a plane, is rather safe (and perhaps uneconomic) substitution of the theoretical (probabilistic) values.

APPENDIX B. EFFECT OF THE RESISTANCE VARIABILITY

In reliability analysis of a generic cross section the coefficient of variability $V_R = 0.15$ and the partial factor γ_R = 1,15 are assumed as an example of a code condition. However, resistance of various structural members made of different materials may have different variability and the partial factor. The coefficient of variability V_R can be expected within a broad range from 0,05 up to almost 0,50 (including uncertainty resistance model). This should be reflected by appropriate value of the partial factor γ_R. Assuming lognormal distribution of R, the partial factor γ_R corresponding to the coefficient of variation V_R can be expressed as

$$\gamma_R = \exp(-1{,}65\ V_R)/\exp(\alpha_E\ \beta\ V_R) = \exp(-1{,}65\ V_R)/\exp(-\ 3{,}04\ V_R) \qquad \text{(B.1)}$$

where $\alpha_E\ \beta\ = -\ 0{,}8 \times 3{,}8 = -\ 3{,}04$ as recommended in EN 1990 [1]. Note that for V_R = 0,10 equation B.1 yields the partial factor γ_R = 1,15.

Figure B.1 shows the variation of the partial factor γ_R with the coefficient of variability V_R together with corresponding reliability index β determined taking into account the partial factor γ_R as a function of V_R given by equation (B.1). In Figure B.1 the combination rule A and a generic cross section are considered only.

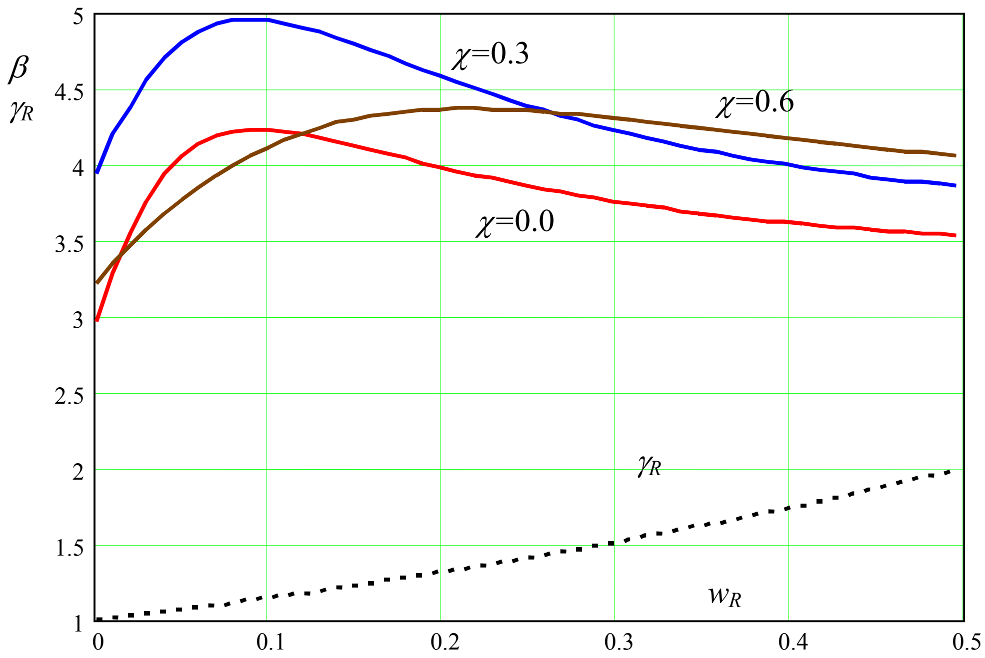


Figure B.1. Variation of the reliability index β of a generic cross section with the coefficient of variability V_R for selected load ratios χ assuming the partial factor γ_R as a function of V_R.

Figure B.1 indicates that if the partial factor γ_R is considered as a function of the coefficient of variability V_R, the effect of resistance variability is not essential. Considering a realistic range of resistance variability $0{,}05 < V_R < 0{,}25$, differences in β values seem to be about 0,5. This finding justifies the concept of a generic cross section used in reliability analysis of alternative load combinations.

APPENDIX C. NOTATION

E	load effect including model uncertainty
E_0	load effect without model uncertainty
E_d	design value of the load effect E
E_k	characteristic value of the load effect E
E_d	design value of the load effect E
G	permanent load including model uncertainty, $G = \theta\, G_0$
G_0	permanent load without load uncertainty
G_d	design value of the resistance G, $G_d=\gamma_G\, G_k$
G_k	characteristic value of the permanent load G
k	load ratio, $k = W_k/Q_k$
P_f	failure probability
Q	main (dominant) variable load including model uncertainty, $Q = \theta\, Q_0$
Q_0	main (dominant) variable load without model uncertainty
Q_d	design value of the variable load Q, $Q_d=\gamma_Q\, Q_k$
Q_k	characteristic value of the variable load G
R	resistance including model uncertainty
R_d	design value of the resistance R, $R_d=\gamma_R\, R_k$
R_k	characteristic value of the resistance R
V_R	coefficient of variation
W_0	main (dominant) variable action without model uncertainty
W	accompanying (non dominant) variable action including model uncertainty, $W = \theta\, W_0$
W_d	design value of the variable load W, $W_d=\gamma_W\, W_k$
W_k	characteristic value of the variable load W
X	vector of basic variables
$Z(X)$	limit state function
α_R	sensitivity factors of R
α_E	sensitivity factors of E
α_G	sensitivity factors of G
α_Q	sensitivity factors of Q
α_W	sensitivity factors of W
β	reliability index, $P_f = \Phi(-\beta)$
$\varphi()$	probability density function
χ	load ratio, $\chi = (Q_k + W_k)/\,(G_k + Q_k + W_k)$
γ_G	partial factor for unfavourable permanent actions G
γ_Q	partial factor for unfavourable variable actions Q
γ_W	partial factor for unfavourable variable actions W
ψ_Q	reduction factor for unfavourable permanent actions Q
ψ_W	reduction factor for unfavourable permanent actions W
ξ	reduction factor for unfavourable permanent actions G
θ	coefficient of model uncertainty
Φ	distribution function of standardised normal distribution

ATTACHMENTS

1. MATLAB function "generic(gR,wr,k)".

Matlab function "Generic" is intended for investigation of the combination rules provided in EN 1990. A general structural member of the resistance R (including model uncertainty) is considered.

Function "Generic" calls function

Action3(mr,sr,skr, Rd,k), which further calls functions

Lnpf (mr,sr,skr,me,se,ske), which calls

Lndens(x,ske,me,se)

Lndist(x,skr,mr,sr)

Ndinv(p)

2. MATLAB function "Action3(mr,sr,skr,Rd,k)"

MATLAB function "Action3" is intended for determining statistical characteristics of the load effect of different combinations of three actions.

Function "Generic" calls function

Lnpf (mr,sr,skr,me,se,ske), which further calls functions

Lndens(x,ske,me,se)

Lndist(x,skr,mr,sr)

Ndinv(p)

3. MATLAB function "Lnpf(mr,sr,skr,me,se,ske)"

MATLAB function "Lnpf" calculates the failure probability using three parameter lognormal distribution for approximation of the load effect and resistance.

Function "Lnpf" further calls functions

Lndens(x,ske,me,se)

Lndist(x,skr,mr,sr)

Ndinv(p)

4. MATLAB function "Lndens(x,mu,sigma,sk)"

MATLAB function "Lndens" is intended for calculation of the probability density function of three-parameter lognormal distribution. The function is called by the function LNPF using command "Lndens(ske,me,se)", and returns the value of probability density function.

5. MATLAB function "Lndist(x,mu,sigma,sk)"

MATLAB function "Lndist" is intended for calculation of the distribution function of three-parameter lognormal distribution. The function is called by the function LNPF using command "Lndist(skr,mr,sr)," and returns the value of the distribution function.

6. MATLAB function "Ndinv(p)"

MATLAB function "Ndinv" calculates the inverse distribution function of the normal distribution (determining the reliability index beta). The function is called by the function Action3 or Action3i using command "Ndinv(p)" (or "Ndinv(p,mu,sigma)" or Ndinv(p,mu)"), and returns the value of the inverse distribution function.

7. MATHCAD sheet"Generic.mcd"

MATHCAD function "Generic" is intended for investigation of the combination rules provided in EN 1990. A general structural member of the resistance r (including model uncertainty) is considered.

8. MATHCAD sheet"Load effect.mcd"

MATHCAD sheet "LoadEffect" is intended for investigation of combination rules provided in EN 1990 by expressions (6.10), (6.10a) and (6.10b) considering three loads: G, Q and W. Turkstra's rule (50-year extremes of a leading and annual extremes of an accompanying action) is applied.

Attachment 1 – MATLAB function "generic(gR,wr,k)"

```
function Gener=Gener(gR,wr,k)
% Program "Generic" is intended for investigation of the combination rules
% provided in EN 1990.
% A generic structural member of the resistance R (including model uncertainty)
% is considered.
% Function "Generic" calls function
%   Action3i(mr,sr,skr, Rd,k,i), which further calls functions
%   Ndinv(p), Lnpf (mr,sr,skr,me,se,ske), which calls
%     Lndens(x,ske,me,se), Lndist(x,skr,mr,sr) and Ndinv(p)
%
% INPUT data describing random variable R: Rd, gR, wR
%
%  VERSION
%  MH, Klokner Institute, Czech Technical University in Prague, 1.08.2003
%
%  Input load ratio k = Wk/Qk used by the function "Action3i",
   % k=0.00; % Input parameter that may be changed
%  Characteristic of the resistance R
   Rd=1; % may be chosen arbitrary
   Rk=Rd*gR; % gR=1.15; gR given by a fixed value not related to wr and beta
%  an alternative is indicated below in the first line of the loop for wr
%
%  Statistical parameters of R (having lognormal distribution) determined
%  in the following loop for selected coefficients of variation wr
   %for i= 3:3 % Range of the loop that may be adjusted.
   %wr=0.05+(i-1)*0.05;  betat = 3.8;
   % gR= exp(0.7*betat*wr)/exp(1.65*wr); Rk=Rd*gR; % alternatively
   Kr=1; wKr=0.05; % lognormal distribution
   mr=Kr*Rk*exp(1.645*wr); wr=(wr^2+wKr^2+wr^2*wKr^2)^0.5;
   sr=mr*wr; skr=3*wr+wr^3;
   % alternatively the mean mr=Rk/(1-1.645*wr)
   Action3(mr,sr,skr,Rd,k) % Call function Action3i
   %end
```

Attachment 2 – MATLAB function "Action3 generic(mr,sr,skr,Rd,k)"

```
function Action3 = action3(mr,sr,skr,Rd,k)
%  CALL
%     Lnpf (mr,sr,skr,me,se,ske), which further calls LNDENS and LNDIST
%  INPUT
%     R: mr,sr,skr; E:me,se,ske; vector of real arguments
%  OUPUT
%     beta: vector of beta values
%  VERSION
%     Klokner Institute, Czech Technical University in Prague, 24.04.2002
%  Laod factors and parameters:
   gG=1.35; gQ=1.5; gW=1.5; psi1=0.7; psi2=0.6; wG=0.1; %psi1=psiQ, psi2=psiW
% Characteristics of variable loads Q and W for k<=(1-psiQ)/(1-psiW)
   if k<=(1-psi1)/(1-psi2);
      mmQ=0.49; wQ=0.4; skQ=1.14; mmW=0.3; wW=0.5; skW=1.14;
      else %Characteristics of variable loads Q and W for k>(1-psiW)/(1-psiQ)
      mmQ=0.2; wQ=1.1; skQ=1.14; mmW=0.7; wW=0.35; skW=1.14;
   end

% Model uncertainties of actions
      Ke=1; wKe=0.00; sKe=wKe*Ke;skKe=3*wKe+wKe^3;
% Parameters k=Wk/Qk given in the function that calls Action3
% Combination factors for expression (6.10) if k<=(1-psiQ)/(1-psiW)=0,75 or  k>(1-
psiW)/(1-psiQ)=0,75
   if k<=(1-psi1)/(1-psi2);
      ksi=1; psiQ=1; psiW=psi2;
      else
      ksi=1; psiQ=psi1; psiW=1;
   end

   %!!!!!!!!!!!!!!!!!!!!!!!!!!Case A, (6.10)
   %Effect of the load ratio CHI for expression (6.10) for the load ratio
CHI=(Qk+Wk/(Gk+Qk+Wk))
for n=1:21 %loop for CHI in the interval <0,1>
   CHI(n)=0+(n-1)*0.0499;
   mG(n)=Rd/(ksi*gG+(CHI(n).*(psiQ*gQ+k*psiW*gW))./((1-CHI(n))*(1+k)));
   sG(n)=mG(n)*wG;
   Qk(n)=CHI(n).*mG(n)./((1-CHI(n))*(1+k)); mQ(n)=Qk(n)*mmQ; sQ(n)=mQ(n)*wQ;
   Wk(n)=Qk(n)*k; mW(n)=Wk(n)*mmW; sW(n)=mW(n)*wW;
   me0(n)=mG(n)+mQ(n)+mW(n); se0(n)=sqrt(sG(n)^2+sQ(n)^2+sW(n)^2);
we0(n)=se0(n)./me0(n);
   ske0(n)=(sQ(n)^3*skQ+sW(n)^3*skW)./se0(n)^3;
   me(n)=(mG(n)+mQ(n)+mW(n))*Ke;
se(n)=me0(n)*Ke.*sqrt(wKe^2+we0(n)^2+we0(n)^2.*wKe^2);

ske(n)=me0(n)^3.*Ke^3.*(wKe^3*skKe+we0(n)^3.*ske0(n)+we0(n)^2*wKe^2*6)./se(n)^3;
   %mG(n).*gG+Qk(n).*gQ+Wk(n).*psiW.*gW
   pf(n)=Lnpf(mr,sr,skr,me(n),se(n),ske(n)); beta(n)= -ndinv(pf(n)); pft(n)=0.0000723;
betat(n)=3.8;
```

```
    sg(n)=sqrt(sr^2+se(n)^2);
    alG(n)=-mG(n)*sqrt(wG^2+wKe^2+wG^2*wKe^2)./sg(n);
    alQ(n)=-mQ(n)*sqrt(wQ^2+wKe^2+wQ^2*wKe^2)./sg(n);
    alW(n)=-mW(n)*sqrt(wW^2+wKe^2+wW^2*wKe^2)./sg(n);
    ale(n)=-se(n)./sg(n);
    alr(n)=sr./sg(n);
end% end of the loop
      % Check selected values
         beta(1), beta(21), %The first and the last Beta
         sg(1);
         alr(1); ale(1); %The first and alr and alr
         sr; se(1);

 %!!!!!!!!!!!!!!!!!!!!!!!!!Case B, (6.10a)
% Combination factors for expression (6.10a) for any k<=>(1-psiW)/(1-psiQ)=0,75

    if k<=(1-psi1)/(1-psi2);
       ksi=1; psiQ=psi1; psiW=psi2;
       else
       ksi=1; psiQ=psi1; psiW=psi2;
    end

for n=1:12 %loop for CHI in the interval <-1,1>
    CHIa(n)=0+(n-1)*0.0499;
    mG(n)=Rd/(ksi*gG+(CHIa(n).*(psiQ*gQ+k*psiW*gW))./((1-CHIa(n))*(1+k)));
    sG(n)=mG(n)*wG;
    Qk(n)=CHIa(n).*mG(n)./((1-CHIa(n))*(1+k)); mQ(n)=Qk(n)*mmQ; sQ(n)=mQ(n)*wQ;
    Wk(n)=Qk(n)*k; mW(n)=Wk(n)*mmW; sW(n)=mW(n)*wW;
    me0(n)=mG(n)+mQ(n)+mW(n); se0(n)=sqrt(sG(n)^2+sQ(n)^2+sW(n)^2);
we0(n)=se0(n)./me0(n);
    ske0(n)=(sQ(n)^3*skQ+sW(n)^3*skW)./se0(n)^3;
    me(n)=(mG(n)+mQ(n)+mW(n))*Ke;
se(n)=me0(n)*Ke.*sqrt(wKe^2+we0(n)^2+we0(n)^2.*wKe^2);

ske(n)=me0(n)^3.*Ke^3.*(wKe^3*skKe+we0(n)^3.*ske0(n)+we0(n)^2*wKe^2*6)./se(n)^3;
    %mG(n).*gG+Qk(n).*gQ+Wk(n).*psiW.*gW
    pfa(n)=Lnpf(mr,sr,skr,me(n),se(n),ske(n)); betaa(n)= -ndinv(pfa(n));
end% end of the loop

%!!!!!!!!!!!!!!!!!!!!!!!!!Case B, (6.10b)
    if k<=(1-psi1)/(1-psi2);
       ksi=0.85; psiQ=1; psiW=psi2;
       else
       ksi=0.85; psiQ=psi1; psiW=1;
    end
for n=1:19 %loop for CHI in the interval <-1,1> for expression (6.10b)
    CHIb(n)=0.1+(n-1)*0.0499;
    mG(n)=Rd/(ksi*gG+(CHIb(n).*(psiQ*gQ+k*psiW*gW))./((1-CHIb(n))*(1+k)));
    sG(n)=mG(n)*wG;
    Qk(n)=CHIb(n).*mG(n)./((1-CHIb(n))*(1+k)); mQ(n)=Qk(n)*mmQ; sQ(n)=mQ(n)*wQ;
```

```
   Wk(n)=Qk(n)*k; mW(n)=Wk(n)*mmW; sW(n)=mW(n)*wW;
   me0(n)=mG(n)+mQ(n)+mW(n); se0(n)=sqrt(sG(n)^2+sQ(n)^2+sW(n)^2);
we0(n)=se0(n)./me0(n);
   ske0(n)=(sQ(n)^3*skQ+sW(n)^3*skW)./se0(n)^3;
   me(n)=(mG(n)+mQ(n)+mW(n))*Ke;
se(n)=me0(n)*Ke.*sqrt(wKe^2+we0(n)^2+we0(n)^2.*wKe^2);

ske(n)=me0(n)^3.*Ke^3.*(wKe^3*skKe+we0(n)^3.*ske0(n)+we0(n)^2*wKe^2*6)./se(n)^3;
   %mG(n).*gG+Qk(n).*gQ+Wk(n).*psiW.*gW
   pfb(n)=Lnpf(mr,sr,skr,me(n),se(n),ske(n)); betab(n)= -ndinv(pfb(n));
end% end of the loop

%!!!!!!!!!!!!!!!!!!!!!!!!!Case C, (6.10amod)
   ksi=1; psiQ=0.0; psiW=0.0;

%Effect of the load ratio CHI for expression (6.10amod)=(6.10c)
for n=1:4 %loop for CHI in the interval <-1,1>
   CHIc(n)=0+(n-1)*0.0499;
   mG(n)=Rd/(ksi*gG+(CHIc(n).*(psiQ*gQ+k*psiW*gW))./((1-CHIc(n))*(1+k)));
   sG(n)=mG(n)*wG;
   Qk(n)=CHIc(n).*mG(n)./((1-CHIc(n))*(1+k)); mQ(n)=Qk(n)*mmQ; sQ(n)=mQ(n)*wQ;
   Wk(n)=Qk(n)*k; mW(n)=Wk(n)*mmW; sW(n)=mW(n)*wW;
   me0(n)=mG(n)+mQ(n)+mW(n); se0(n)=sqrt(sG(n)^2+sQ(n)^2+sW(n)^2);
we0(n)=se0(n)./me0(n);
   ske0(n)=(sQ(n)^3*skQ+sW(n)^3*skW)./se0(n)^3;
   me(n)=(mG(n)+mQ(n)+mW(n))*Ke;
se(n)=me0(n)*Ke.*sqrt(wKe^2+we0(n)^2+we0(n)^2.*wKe^2);

ske(n)=me0(n)^3.*Ke^3.*(wKe^3*skKe+we0(n)^3.*ske0(n)+we0(n)^2*wKe^2*6)./se(n)^3;
   %mG(n).*gG+Qk(n).*gQ+Wk(n).*psiW.*gW
   pfc(n)=Lnpf(mr,sr,skr,me(n),se(n),ske(n)); betac(n)= -ndinv(pfc(n));
end% end of the loop


%!!!!!!!!!!!!!!!!!!!!!!!!!Case D CSN (6.10)
gG=1.2; gQ=1.4; gW=1.4; %alternative values og gamma
   if k<=(1-psi1)/(1-psi2);
      ksi=1; psiQ=1; psiW=psi2;
      else
      ksi=1; psiQ=psi1; psiW=1;
   end
   %Effect of the load ratio CHI for expression (6.10)
   for n=1:21 %loop for CHI in the interval <0,1>
   CHI(n)=0+(n-1)*0.0499;
   mG(n)=Rd/(ksi*gG+(CHI(n).*(psiQ*gQ+k*psiW*gW))./((1-CHI(n))*(1+k)));
   sG(n)=mG(n)*wG;
   Qk(n)=CHI(n).*mG(n)./((1-CHI(n))*(1+k)); mQ(n)=Qk(n)*mmQ; sQ(n)=mQ(n)*wQ;
   Wk(n)=Qk(n)*k; mW(n)=Wk(n)*mmW; sW(n)=mW(n)*wW;
   me0(n)=mG(n)+mQ(n)+mW(n); se0(n)=sqrt(sG(n)^2+sQ(n)^2+sW(n)^2);
we0(n)=se0(n)./me0(n);
```

```
   ske0(n)=(sQ(n)^3*skQ+sW(n)^3*skW)./se0(n)^3;
   me(n)=(mG(n)+mQ(n)+mW(n))*Ke;
se(n)=me0(n)*Ke.*sqrt(wKe^2+we0(n)^2+we0(n)^2.*wKe^2);

ske(n)=me0(n)^3.*Ke^3.*(wKe^3*skKe+we0(n)^3.*ske0(n)+we0(n)^2*wKe^2*6)./se(n)^3;
   %mG(n).*gG+Qk(n).*gQ+Wk(n).*psiW.*gW
   pfd(n)=Lnpf(mr,sr,skr,me(n),se(n),ske(n)); betad(n)= -ndinv(pfd(n));
end% end of the loop


%!!!!!!!!!!!!!!!!!!!!!!!!!!Case E: BSI (6.10)
ksi=1; psiQ=1; psiW=1; %reduction factor not considered
% Combination factors for expression (6.10) for one k=0, or two k>0
    Rd=Rd*1.15/1.1; % adjustement for different for gm=1.10;
if k<=0.001;
      gG=1.4; gQ=1.6; gW=1.4;
      else
       gG=1.2; gQ=1.2; gW=1.2;
   end
   %Effect of the load ratio CHI for expression (6.10)
   for n=1:21 %loop for CHI in the interval <0,1>
   CHI(n)=0+(n-1)*0.0499;
   mG(n)=Rd/(ksi*gG+(CHI(n).*(psiQ*gQ+k*psiW*gW))./((1-CHI(n))*(1+k)));
   sG(n)=mG(n)*wG;
   Qk(n)=CHI(n).*mG(n)./((1-CHI(n))*(1+k)); mQ(n)=Qk(n)*mmQ; sQ(n)=mQ(n)*wQ;
   Wk(n)=Qk(n)*k; mW(n)=Wk(n)*mmW; sW(n)=mW(n)*wW;
   me0(n)=mG(n)+mQ(n)+mW(n); se0(n)=sqrt(sG(n)^2+sQ(n)^2+sW(n)^2);
we0(n)=se0(n)./me0(n);
   ske0(n)=(sQ(n)^3*skQ+sW(n)^3*skW)./se0(n)^3;
   me(n)=(mG(n)+mQ(n)+mW(n))*Ke;
se(n)=me0(n)*Ke.*sqrt(wKe^2+we0(n)^2+we0(n)^2.*wKe^2);

ske(n)=me0(n)^3.*Ke^3.*(wKe^3*skKe+we0(n)^3.*ske0(n)+we0(n)^2*wKe^2*6)./se(n)^3;
   %mG(n).*gG+Qk(n).*gQ+Wk(n).*psiW.*gW
   pfe(n)=Lnpf(mr,sr,skr,me(n),se(n),ske(n)); betae(n)= -ndinv(pfe(n));
end% end of the loop

% Ploting beta and pf and alpha versus CHI

ksi=0.85; psiQ=psi1; psiW=psi2;gG=1.35;gQ=1.5;gW=1.5;
   if k<=(1-psi1)/(1-psi2);
      a=1; b=psi2;
   else
      a=psi1; b=1;
   end
CHIlim=gG*(1-ksi)*(1+k)/(gG*(1-ksi)*(1+k)+gQ*(a-psiQ)+gW*k*(b-psiW))
CHIlimc=gG*(1-ksi)*(1+k)/(gG*(1-ksi)*(1+k)+gQ*a+gW*k*b)
   for n=1:14
   B(n)=3+(n-1)*0.15;
   C(n)=10^(-2-(n-1)*0.35);
```

```
    end
    maxbeta=ceil(max(beta));
    maxpf=ceil(100*max(pf))/100;
    h=figure(1);

% plot beta versus CHI
    subplot(2,1,1)%CSN Beta: ,CHI,betad,'r.', BSI beta CHI,betae,'k--'
    plot(CHI,beta,'r',CHI,betae,'k--',CHIa,betaa,'b',CHIb,betab,'b',CHIc,betac,'g--',CHI,betat,'k-
.',CHIlimc,B,'g.',CHIlim,B,'b.','LineWidth',1.5,'MarkerSize',5) %betaa,CHI,betab,CHI,betat,
    grid,axis([0,1,3,6]) % maxbeta
    ylabel('Index Beta')
    subplot(2,1,2)
% plot pf versus CHI
    semilogy(CHI,pf,'r',CHI,pfe,'k--',CHIa,pfa,'b',CHIb,pfb,'b',CHIc,pfc,'g--',CHI,pft,'k-
.',CHIlimc,C,'g.',CHIlim,C,'b.','LineWidth',1.5,'MarkerSize',5)
    %plt using semilogarthmical scale, ,pfa,CHI,pfb,CHI
    grid,axis([0,1,1e-8,maxpf])
    ylabel('Probability Pf')

    %subplot(3,1,3) % not generally active
% plot Alphas versus CHI
    %plot(CHI,alG,CHI,alQ,CHI,alW,CHI,alr,CHI,ale,
    %'LineWidth',1.5) %Alphasplt scale, ,pfa,CHI,pfb,CHI
    %grid,axis([0,1,-1,1])
    %xlabel('Load ratio CHI=(Qk+Wk)/(Gk+Qk+Wk)')
    %ylabel('Alphas for A - (6.10)')
```

Attachment 3 – MATLAB function "Lnpf(mr,sr,skr,me,se,ske)"

```
function pf = Lnpf (mr,sr,skr,me,se,ske)
%  DESCRIPTION, 19.09.1999
%     Lnpf evaluates the probability of failure pf considering the fundamental
%     limit state function G = R - E.
%  CALL
%     val = Lnpf (mr,sr,skr,me,se,ske);
%  FUNCTIONS USED
%     lndens(x,ske,me,se).*Lndist(x,skr,mr,sr)
%  INPUT
%     mr    : the mean of R
%     sr    : standard deviation of R
%     skr   : coefficient of skewness of R (must be given)
%     me    : the mean of E
%     se    : standard deviation of E
%     ske   : coefficient of skewness of E (must be given)
%  OUPUT
%     val   : failure probability pf
%  VERSION
%     MH, Czech Technical University in Prague, Klokner Institute, 2.8.2003
%  Initialization
cr=(0.5*skr+(skr^2/4+1)^0.5)^(1/3)-(-0.5*skr+(skr^2/4+1)^0.5)^(1/3);%constant of
lognormal R
ce=(0.5*ske+(ske^2/4+1)^0.5)^(1/3)-(-0.5*ske+(ske^2/4+1)^0.5)^(1/3);%constant of
lognormal E
if cr==0
  r0=10^10;
else
  r0=mr-sr/cr;        %bound of lognormal distribution of R
end
if ce==0
  e0=10^10;
else
  e0=me-se/ce;        %bound of lognormal distribution of E
end
% Determination of integration interval
k=10; % Coefficient of standard deviation
if skr>0;          %R positive
  if ske>0;        %R and E positive
    a=max(r0,e0);
    b=min(mr+k*sr,me+k*se);     % limit 6*sr
  else
    if ce==0      %R pos, E sym
      a=max(r0,me-k*se);
      b=min(mr+k*sr,me+k*se);
    else          % R pos E neg
      a=max(r0,me-k*se);
      b=min(e0,mr+k*sr);
    end
```

```
  end
else              % R sym or neg
  if cr==0;        % R sym c=0 (sk=0)
    if ce==0      % both R and E sym
      a=max(mr-k*sr,me-k*se);
      b=min(mr+k*sr,me+k*se);
    else
      if ske>0    %R sym, E pos
        a=max(mr-k*sr,e0);
        b=min(mr+k*sr,me+k*se);
      else        % R sym, E neg
        a=max(mr-k*sr,me-k*se);
        b=min(mr-k*sr,e0);
      end
    end
  else            % R neg
    if ske>0;     % R neg, E pos
      a=max(mr-k*sr,e0);
      b=min(r0,me+k*se);
    else          % E sym or neg
      if ce==0    % R neg, E sym
        a=max(mr-k*sr,me-k*se);
        b=min(r0,me+k*se);
      else        % R neg, E neg
        a=max(mr-k*sr,me-k*se);
        b=min(e0,r0);
      end
    end
  end
end
% Integration interval
if a<0
  a=0;
   end
  delta=b-a;
  a=a+0.00000001*delta; % adjusted integration limits to avoid singularity
  b=b-0.00000001*delta;
  if a>b
  '  error in input data, a>b, increase the lower limit of the design parameter'
  pause
else
   n=20; inc=(b-a)/n; Y=0; % parameters of integration, n may adjusted if needed
for x=a:inc:b; % Integration of failure probability by trapezoidal rule
   e= Lndens(x,me,se,ske); % call Lndens
   r= Lndist(x,mr,sr,skr); % call Lndist
   Y=Y+ e*r; % sum of e*r
end
   pf=inc*Y;
end
```

Attachment 4 – MATLAB function"Lndens(x,mu,sigma,sk)"

```
function val = Lndens (x,mu,sigma,sk)
%  DESCRIPTION
%     NDENS evaluates the one-dimensional normal density function.
%  CALL
%     val = Lndens (x,sk);
%     val = Lndens (x,sk,mu,sigma);
%  INPUT
%     x     : real vector of arguments
%     sk    : coefficient of skewness (must be given)
%     mu    : mean value; optional; default = 0.0 (i.e. standard)
%     sigma : std. dev. > 0; optional; default = 1.0 (i.e. standard)
%  OUPUT %     val: vector of normal density values for the x's
%  VERSION %     Milan Holicky, Czech Technical University in Prague, Klokner Institute
%     18.09.1999
if nargin < 3
  mu    = 0.0;
  sigma = 1.0;
end
%  Evaluate
x   = (x-mu)/sigma;                        % normalize
c=(0.5*sk+(sk^2/4+1)^0.5)^(1/3)-(-0.5*sk+(sk^2/4+1)^0.5)^(1/3);%constant of lognormal
if c==0; % sk=0
    x0=10^10;
  else
    x0=-1/c; %bound of the distribution
  end
if sk>0; %check of x range
  if x0>x
    error ('x out of range')
  else
  end
else
  if x0<x
    error('x out of range')
  else
  end
end
%
if abs(c)>0.0001; %if for c=0 (sk=0)
    tt=sign(sk)*(log(abs(x+1/c))+log(abs(c))+0.5*log(1+c^2))/((log(1+c^2))^0.5);
  else
    tt=x;
  end
if abs(c)>0.0001; %if for c=0 (sk=0)
  val = exp(-0.5*tt.^2)/(sqrt(2*pi)*sigma*abs(x+1/c)*(log(1+c^2))^0.5);
else
  val = exp(-0.5*tt.^2)/(sigma*sqrt(2*pi));
end
```

Attachment 5 – MATLAB function"Lndist(x,mu,sigma,sk)"

```
function val = Lndist (x,mu,sigma,sk)
% DESCRIPTION, 18.09.1999
%    LNDIST evaluates the one-dimensional lognormal distribution function.
% CALL
%    val = lndist (x,sk)
%    val = lndist (x,sk,mu,sigma)
% INPUT
%    x     : real vector of arguments.
%    sk    : coefficient of skewness; default = 0.0 (i.e. normal distribution)
%    mu    : mean value; optional; default = 0.0 (i.e. standard).
%    sigma : std. dev. > 0; optional; default = 1.0 (i.e. standard).
% OUPUT
%    val   : vector of the lognormal distribution evaluated at the x's.
% VERSION
%    Milan Holicky, Czech Technical University in Prague, Klokner Institute
% Initialization
if nargin < 3
  mu    = 0.0;
  sigma = 1.0;
end
x   = (x-mu)/sigma;            % standardize
c=(0.5*sk+(sk^2/4+1)^0.5)^(1/3)-(-0.5*sk+(sk^2/4+1)^0.5)^(1/3);%constant of lognormal
if c==0; % sk=0
    x0=10^10;
  else
    x0=-1/c; %bound of the distribution
  end
if sk>0; %check of x range
  if x0>x
    error ('x out of range')
  else
  end
else
  if x0<x
    error('x out of range')
  else
  end
end
if abs(c)>0.01; %if for c=0 (sk=0)
    tt=sign(sk)*(log(abs(x+1/c))+log(abs(c))+0.5*log(1+c^2))/((log(1+c^2))^0.5);
  else
    tt=x;
end
val = (1+erf(tt/sqrt(2)))/2;    % transformed error function
```

Attachment 6 – MATLAB function"Ndinv(x,mu,sigma)"

```
function [x] = norminv(p,mu,sigma)
%NORMINV Inverse of the normal cumulative distribution function (cdf).
%   X = NORMINV(P,MU,SIGMA) returns the inverse cdf for the normal
%   distribution with mean MU and standard deviation SIGMA, evaluated at
%   the values in P.
%   Default values for MU and SIGMA are 0 and 1, respectively.
%
%   MH, Klokner Institute, CTU Prague 4.8.2003
%
if nargin < 2
    mu = 0;
end
if nargin < 3
    sigma = 1;
end
% Return NaN for out of range parameters or probabilities.
    sigma(sigma <= 0) = NaN;
    p(p < 0 | 1 < p) = NaN;
    x0 = -sqrt(2).*erfcinv(2*p);
    x = sigma.*x0 + mu;
```

Attachment 7 – MATHCAD sheet"Generic.Mcd"

Mathcad sheet "Generic" is intended to investigate combination rules provided in EN 1990 by expressions (6.10), (6.10a) and (6.10b) considering a generic structural memeber. Turkstra's rule (50 years extremes of a leading and annual extremes for accompnying action) is applied forthe reference period of 50 years. MH, August 2002.

Design expression: R0k / γM = (ξ) γG*Gk + (γQ)γQ*Qk +(γW)*ψW*Wk
Limit state function: g(X) = ρ * R0 – θ * (G + Q + W), R0=K*fu

Resistance of an element R = ρ * R0 is described by two parameter lognormal distribution LN(μ_R,σ_R), basic variables R0 by LN(ωRk, 0.06μR), ω =1/(1 – 2*wR0) and ρ by LN(1.1, 0.05).

Load effect E = θ * (G + Q + W) is discribed by a three parameter lognormal distribution LNα(μ_E,σ_E,α_E), permanent load G by N(Gk, 0.1*Gk), 50 years imposed load Q by GUM(0.6 Qk, 0.35 μQ), annual wind load W by GUM(0.3Wk, 0.5μW), uncertainty θ by LN(1.0, 0.05).

Parameters: χ=(Qk+Wk)/(Gk+Qk+Wk), k=Wk/Qk, factors γm, γG, γQ, γW, ξ, ψQ, ψW.

1 Input data:

Constants: $\xi := 0.85$ $\quad \psi Q := 0.70$ $\quad \psi W := 0.60$ $\quad k := 0.0$ $\quad Rd := 1$

Range variables: $\chi := 0, 0.09 .. 0.99$ $\quad \gamma m := 1.0, 1.05 .. 1.5$ $\quad \gamma G := 1.1, 1.15 .. 1.5$ $\quad \gamma Q := 1.2, 1.25 .. 1.6$ $\quad \gamma W := 1.2, 1.25 .. 1.6$

Load parameters:

Normal distribution of G: μG = Gk $\quad wG := 0.1$

Gumbel distribution of Q: μQ = mQ*Qk $\quad mQ := 0.6$ $\quad wQ := 0.35$ $\quad \alpha Q := 1.14$

Gumbel distribution of W: μW = mW*Wk $\quad mW := 0.3$ $\quad wW := 0.5$ $\quad \alpha W := 1.14$

Model uncertainty θ, Lognormal distribution: $\mu\theta := 1.0$ $\quad w\theta := 0.05$ $\quad \alpha\theta := 3 \cdot w\theta + w\theta^3$

Resistance varables: μR0 = ω*R0k: $\quad wR0 := 0.0, 0.12 .. 0.35$ $\quad \omega(wR0) := \exp(1.65 \cdot wR0)$ $\quad \alpha R0(wR0) := wR0 \cdot 3 + wR0^3$

Model uncertainty ρ, Lognormal distribution: $\mu\rho := 1.0$ $\quad w\rho := 0.05$ $\quad \alpha\rho := 3 \cdot w\rho + w\rho^3$

2 Determination of the load variables G, Q and W for a given resistance Rd

Characteristic values Gk=μG, Qk and Wk determined assuming Ed=Rd. Thus γG*Gk+γQ*Qk+γW*Wk=Rd

$$\mu G(\chi, \gamma G, \gamma Q, \gamma W) := \frac{Rd}{\left[\gamma G + \frac{(\gamma Q + k \cdot \gamma W) \cdot \chi}{(1 + k) \cdot (1 - \chi)}\right]}$$

Check: $\boxed{\mu G(0.5, 1.35, 1.5, 1.5) = 0.351}$

$$Qk(\chi, \gamma G, \gamma Q, \gamma W) := \frac{\chi \, \mu G(\chi, \gamma G, \gamma Q, \gamma W)}{(1 + k) \cdot (1 - \chi)} \qquad Wk(\chi, \gamma G, \gamma Q, \gamma W) := k \cdot Qk(\chi, \gamma G, \gamma Q, \gamma W)$$

$\boxed{Qk(0.5, 1.35, 1.5, 1.5) = 0.351}$

$$Ed(\chi, \gamma G, \gamma Q, \gamma W) := \gamma G \cdot \mu G(\chi, \gamma G, \gamma Q, \gamma W) + \gamma Q \cdot Qk(\chi, \gamma G, \gamma Q, \gamma W) + \gamma W \cdot Wk(\chi, \gamma G, \gamma Q, \gamma W)$$

$\boxed{Wk(0.5, 1.35, 1.5, 1.5) = 0}$

Normal distribution of G: $\sigma G(\chi, \gamma G, \gamma Q, \gamma W) := wG \cdot \mu G(\chi, \gamma G, \gamma Q, \gamma W)$

$\boxed{Ed(0.5, 1.35, 1.5, 1.5) = 1}$

Gumbel distribution of Q: $\mu Q(\chi, \gamma G, \gamma Q, \gamma W) := mQ \cdot Qk(\chi, \gamma G, \gamma Q, \gamma W)$ $\quad \sigma Q(\chi, \gamma G, \gamma Q, \gamma W) := wQ \cdot \mu Q(\chi, \gamma G, \gamma Q, \gamma W)$

Gumbel distribution of W: $\mu W(\chi, \gamma G, \gamma Q, \gamma W) := mW \cdot Wk(\chi, \gamma G, \gamma Q, \gamma W)$ $\quad \sigma W(\chi, \gamma G, \gamma Q, \gamma W) := wW \cdot \mu W(\chi, \gamma G, \gamma Q, \gamma W)$

3 Load effect E = θ * (G+Q+W) =θ*E0:

The mean and st. deviation of E0: $\mu E0(\chi, \gamma G, \gamma Q, \gamma W) := \mu G(\chi, \gamma G, \gamma Q, \gamma W) + \mu Q(\chi, \gamma G, \gamma Q, \gamma W) + \mu W(\chi, \gamma G, \gamma Q, \gamma W)$

$$\sigma E0(\chi, \gamma G, \gamma Q, \gamma W) := \sqrt{wG^2 \cdot \mu G(\chi, \gamma G, \gamma Q, \gamma W)^2 + wQ^2 \cdot \mu Q(\chi, \gamma G, \gamma Q, \gamma W)^2 + wW^2 \cdot \mu W(\chi, \gamma G, \gamma Q, \gamma W)^2}$$

The coefficient of variation of E0 (without model uncertainty θ): $\mu E0(0.4, 1.35, 1.5, 1.5) = 0.596$

$$wE0(\chi,\gamma G,\gamma Q,\gamma W) := \frac{\sigma E0(\chi,\gamma G,\gamma Q,\gamma W)}{\mu E0(\chi,\gamma G,\gamma Q,\gamma W)} \quad \mu E(\chi,\gamma G,\gamma Q,\gamma W) := \mu\theta\cdot\mu E0(\chi,\gamma G,\gamma Q,\gamma W) \quad wE0(0.4, 1.35, 1.5, 1.5) = 0.123$$

The coefficient of variation of E: $wE(\chi,\gamma G,\gamma Q,\gamma W) := \sqrt{wE0(\chi,\gamma G,\gamma Q,\gamma W)^2 + w\theta^2 + wE0(\chi,\gamma G,\gamma Q,\gamma W)^2\cdot w\theta^2}$

The standard deviation of E: $\sigma E(\chi,\gamma G,\gamma Q,\gamma W) := \mu E(\chi,\gamma G,\gamma Q,\gamma W)\cdot wE(\chi,\gamma G,\gamma Q,\gamma W)$

4 Three parameter lognormal distribution of E

$wE(0.4, 1.35, 1.5, 1.5) = 0.133$

Skewness of E0:

$$\alpha E0(\chi,\gamma G,\gamma Q,\gamma W) := \frac{\sigma Q(\chi,\gamma G,\gamma Q,\gamma W)^3\cdot\alpha Q + \sigma W(\chi,\gamma G,\gamma Q,\gamma W)^3\cdot\alpha W}{\sigma E0(\chi,\gamma G,\gamma Q,\gamma W)^3}$$

$\boxed{\alpha E0(0.4, 1.35, 1.5, 1.5) = 0.614}$

Skewness of E:

$$\alpha E(\chi,\gamma G,\gamma Q,\gamma W) := \frac{wE0(\chi,\gamma G,\gamma Q,\gamma W)^3\cdot\alpha E0(\chi,\gamma G,\gamma Q,\gamma W) + 6\cdot w\theta^2\cdot wE0(\chi,\gamma G,\gamma Q,\gamma W)^2 + w\theta^3\cdot\alpha\theta}{wE(\chi,\gamma G,\gamma Q,\gamma W)^3}$$

$\boxed{\alpha E(0.4, 1.35, 1.5, 1.5) = 0.591}$

Parameter C:

$$C(\chi,\gamma G,\gamma Q,\gamma W) := \frac{\left(\sqrt{\alpha E(\chi,\gamma G,\gamma Q,\gamma W)^2 + 4} + \alpha E(\chi,\gamma G,\gamma Q,\gamma W)\right)^{\frac{1}{3}} - \left(\sqrt{\alpha E(\chi,\gamma G,\gamma Q,\gamma W)^2 + 4} - \alpha E(\chi,\gamma G,\gamma Q,\gamma W)\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}}$$

Parameters of transformed variable:

$$mE(\chi,\gamma G,\gamma Q,\gamma W) := -\ln(|C(\chi,\gamma G,\gamma Q,\gamma W)|) + \ln(\sigma E(\chi,\gamma G,\gamma Q,\gamma W)) - (0.5)\cdot\ln\left(1 + C(\chi,\gamma G,\gamma Q,\gamma W)^2\right)$$

$$sE(\chi,\gamma G,\gamma Q,\gamma W) := \sqrt{\ln\left(1 + C(\chi,\gamma G,\gamma Q,\gamma W)^2\right)} \quad x0(\chi,\gamma G,\gamma Q,\gamma W) := \mu E(\chi,\gamma G,\gamma Q,\gamma W) - \frac{1}{C(\chi,\gamma G,\gamma Q,\gamma W)}\sigma E(\chi,\gamma G,\gamma Q,\gamma W)$$

Probability density of E, approximation by three parameter lognormal distribution: $\boxed{x0(0.4, 1.35, 1.5, 1.5) = 0.189}$

$$Eln(x,\chi,\gamma G,\gamma Q,\gamma W) := dlnorm(x - x0(\chi,\gamma G,\gamma Q,\gamma W), mE(\chi,\gamma G,\gamma Q,\gamma W), sE(\chi,\gamma G,\gamma Q,\gamma W))$$

5 Resistance variables R = ρ*R_0:

$\mu R(\gamma m, wR0) := \mu\rho\cdot Rd\cdot\gamma m\cdot\omega(wR0)$ Check: $\boxed{\mu R(1.15, 0.15) = 1.473}$

$wR(wR0) := \sqrt{wR0^2 + w\rho^2 + wR0^2\cdot w\rho^2}$ $\quad\sigma R(\gamma m, wR0) := wR(wR0)\,\mu R(\gamma m)$ $\quad\boxed{wR(0.15) = 0.158}$

6 Two parameter lognormal distribution of R

Transformed variable: $mR(\gamma m, wR0) := \ln(\mu R(\gamma m, wR0)) - (0.5)\cdot\ln\left(1 + wR(wR0)^2\right)$ $\quad sR(\gamma m, wR0) := \sqrt{\ln\left(1 + wR(wR0)^2\right)}$

Distribution function $Rln(x,\gamma m, wR0) := plnorm(x, mR(\gamma m, wR0), sR(\gamma m, wR0))$ $\quad mR(1, 015) = 22.038$ $\quad sR(1, 0.15) = 0.157$

7 Failure probability and reliability index β: $d(\chi,\gamma G,\gamma Q,\gamma W) := \mathrm{if}(x0(\chi,\gamma G,\gamma Q,\gamma W) \le 0, 0, x0(\chi,\gamma G,\gamma Q,\gamma W))$

$$pf50(\chi,\gamma m, wR0,\gamma G,\gamma Q,\gamma W) := \int_{d(\chi,\gamma G,\gamma Q,\gamma W)}^{\infty} Eln(x,\chi,\gamma G,\gamma Q,\gamma W)\, Rln(x,\gamma m, wR0)\, dx$$

$\boxed{d(0.4, 1.35, 1.5, 1.5) = 0.189}$

$\boxed{pf50(0.0, 1.15, 0.15, 1.35, 1.5, \psi W\cdot 1.5) = 1.645\times 10^{-4}}$

$$\beta ln(\chi,\gamma m, wR0,\gamma G,\gamma Q,\gamma W) := -qnorm(pf50(\chi,\gamma m, wR0,\gamma G,\gamma Q,\gamma W), 0, 1)$$

$\boxed{\beta ln(0.0, 1.15, 0.15, 1.35, 1.5, \psi W\cdot 1.5) = 3.591}$

8 Reliability index β versus ratio χ: limit for dominant action: $k0 := \dfrac{1-\psi Q}{1-\psi W}$ Check: $\boxed{k0 = 0.75}$

$a := \mathrm{if}(k \le k0, 1, \psi Q)$ $b := \mathrm{if}(k > k0, 1, \psi W)$ $\boxed{a = 1}$ $\boxed{b = 0.6}$

Limit value of χ for (6.10a) and (6.10b)
$$\chi\chi(\gamma G,\gamma Q,\gamma W) := \frac{\gamma G\cdot(1-\xi)(1+k)}{\gamma G\cdot(1-\xi)(1+k) + [\gamma Q\cdot(a-\psi Q) + \gamma W\cdot k\cdot(b-\psi W)]}$$

Limit of χ for (6.10a-mod) and (6.10b)
$$\chi\chi a(\gamma G,\gamma Q,\gamma W) := \frac{\gamma G\cdot(1-\xi)(1+k)}{\gamma G\cdot(1-\xi)(1+k) + (\gamma Q\cdot a + \gamma W\cdot k\cdot b)}$$

$\chi la := \chi\chi a(1.35, 1.5, 1.5)$

$\chi l := \chi\chi(1.35, 1.5, 1.5)$

Target probability $\beta t := 3.8$ Auxiliary: $\chi 0 := 3, 3.5 .. 5$

$\chi a := 0, 0.05 .. \chi l + 0.05$ $\chi b := \chi la - 0.01, \chi la + 0.04 .. 0.999$ $\chi am := 0, 0.05 .. \chi la + 0.04$ Check:

$\chi c := 0, 0.05 .. \chi la + 0.04$ $\boxed{\beta ln(0, 1.15, 0.15, 1.35, a\cdot 1.5, b\cdot 1.5) = 3.591}$ $\boxed{\beta ln(0.999, 1.15, 0.15, 1.35, a\cdot 1.5, b\cdot 1.5) = 3.583}$

Turkstra's for 50 years: $wG = 0.1$ $mQ = 0.6$ $wQ = 0.35$ $mW = 0.3$ $wW = 0.5$ $k = 0$ $\chi l = 0.31$ $\chi la = 0.119$

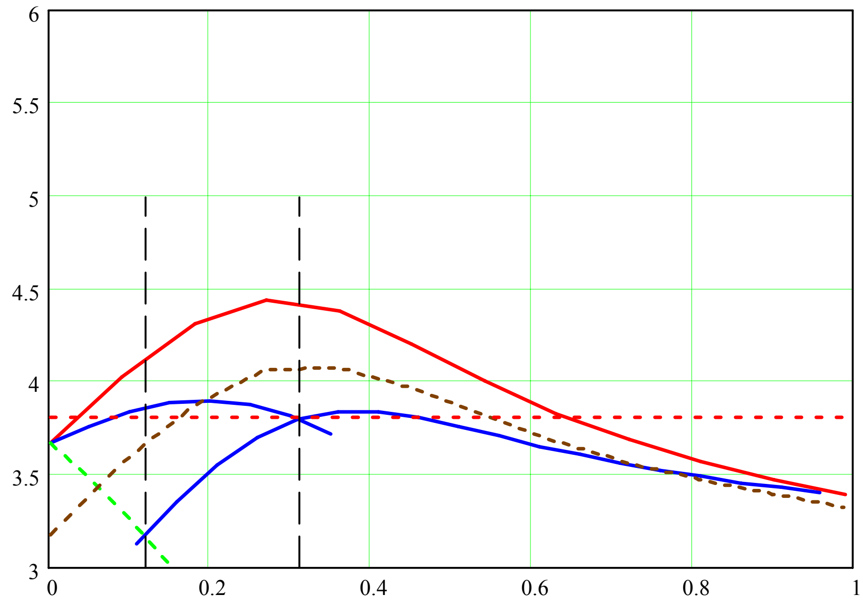


Figure 1: Reliability index β corresponding to equation (6.10), (6.10a) and (6.10b).

Note that expression (6.10) is represented by solid red line, expressions (6.10a) and (6.10b) by solid blue line,expression (6.10a,mod) by dashed green line, BSI combination by dashed brown line, CSN combination by red dashed line, target reliability index by horizontal black dashed line. The twin expressions (6.10a) and 6.10b) provide the most uniform distribution of the reliability index β against the load ratio χ with reliability index greater than the target value β = 3,8 for majority of χ.

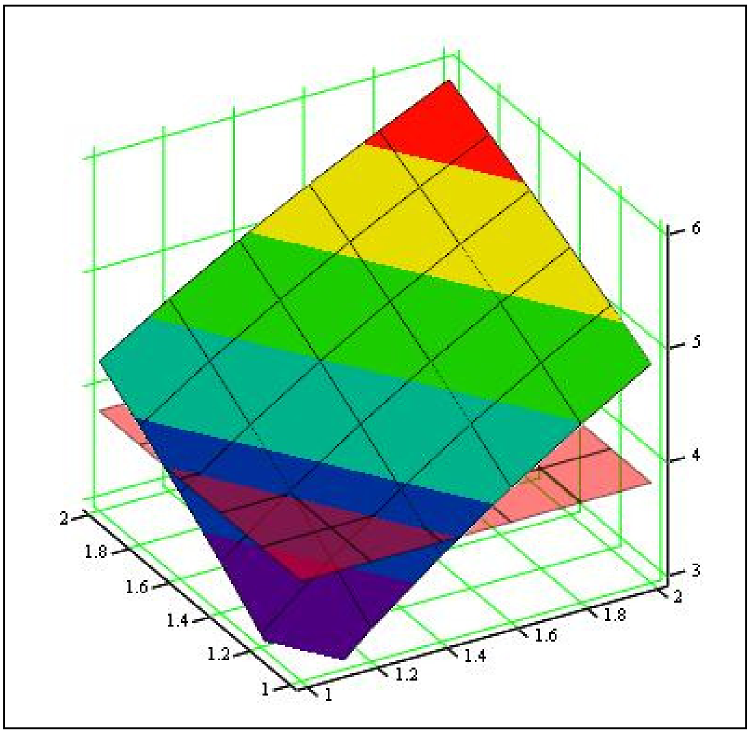
9 Reliability index β versus γG a γQ:

$\beta 1(\gamma G, \gamma Q) := \beta \ln(0.4, 1.15, 0.15, \gamma G, a \cdot \gamma Q, b \cdot \gamma Q)$ $\quad \beta 1t(\gamma G, \gamma Q) := 3.8$

Figure 2: Reliability index β corresponding to expression (6.10) versus partial factors γG a γQ.

Parameters : $k = 0$

$mQ = 0.6$ $\quad wQ = 0.35$ $\quad mW = 0.3$ $\quad wW = 0.5$



$\beta 1, \beta 1t$

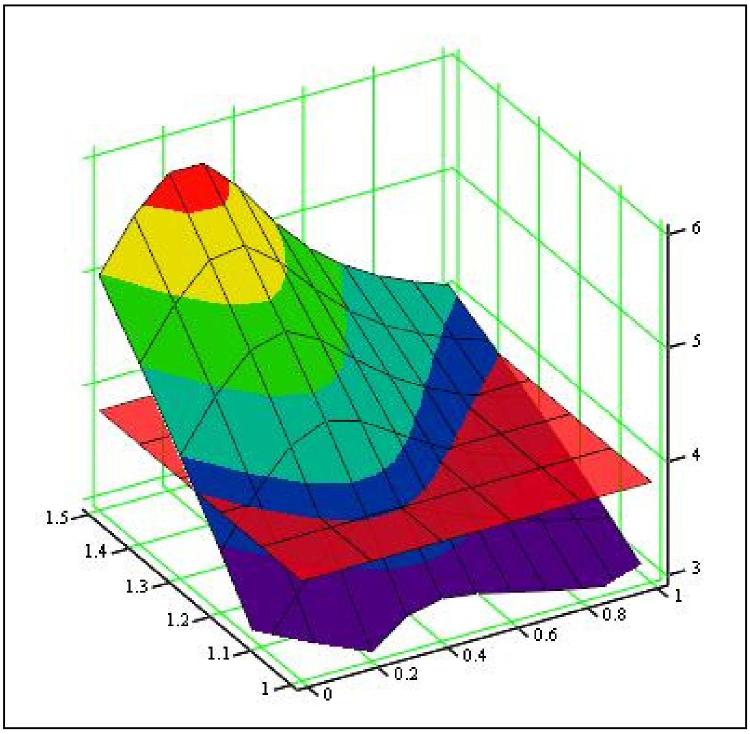
10 Reliability index β versus γm a χ:

$\beta 2(\chi, \gamma m) := \beta \ln(\chi, \gamma m, 0.10, 1.2, a \cdot 1.4, b \cdot 1.4)$ $\quad \beta 2t(\chi, \gamma m) := 3.8$

Figure 3: Reliability index β corresponding to expression (6.10) versus partial factor γm and parameter χ.

Parameters : $k = 0$

$mQ = 0.6$ $\quad wQ = 0.35$ $\quad mW = 0.3$ $\quad wW = 0.5$



$\beta 2, \beta 2t$

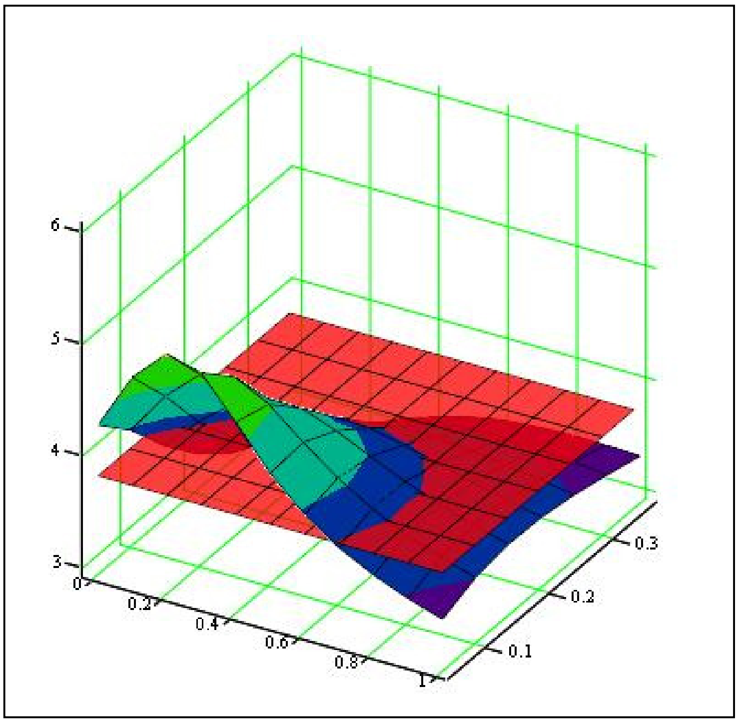
11 Reliability index β versus γm a χ:

$\beta 2(\chi, wR0) := \beta ln(\chi, 1.15, wR0, 1.35, a \cdot 1.5, b \cdot 1.5)$ $\beta 2t(\chi, wR0) := 3.8$

Figure 4: Reliability index β corresponding to expression (6.10) versus parameter χ and the coefficient of variation *wR*.

Parameters : $k = 0$

$mQ = 0.6$ $wQ = 0.35$ $mW = 0.3$ $wW = 0.5$



$\beta 2, \beta 2t$

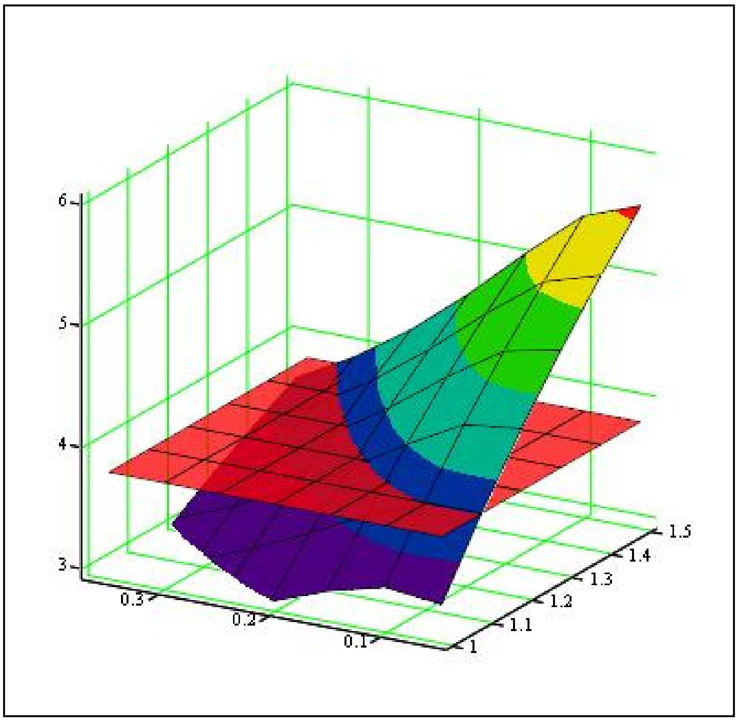
12 Reliability index β versus γm a wR:

$\beta 4(\gamma m, wR0) := \beta ln(0.4, \gamma m, wR0, 1.2, a \cdot 1.4, b \cdot 1.4)$ $\beta 4t(\gamma m, wR0) := 3.8$

Figure 5: Reliability index β corresponding to expression (6.10) versus parameter γR and the coefficient of variation *wR*.

Parameters : $k = 0$

$mQ = 0.6$ $wQ = 0.35$ $mW = 0.3$ $wW = 0.5$



$\beta 4, \beta 4t$

Attachment 8 – MATHCAD sheet"LoadEffect.mcd"

MATHCADsheet "LoadEfect"

Mathcad sheet "LoadEffect" is intended for investigation of combination rules provided in EN 1990 by expressions (6.10), (6.10a) and (6.10b) considering three loads: G, Q and W. Turkstra's rule (50 years extremes of a leading and annual extremes of an accompnying action) is applied. MH, May 2003.

Design Load effect Ed = (ξ)γG*Gk + (γQ)γQ*Qk +(γ\

Stochastic model E = θ * (G + Q + W)

Load effect E = θ * (G + Q + W) is discribed by a three parameter lognormal distribution LNα (μ_E,σ_E,α_E), permanent load G by N(Gk, 0.1*Gk), 50 yearsimposed load Q by GUM(0.6 Qk, 0.35 μQ), annual wind load W by GUM(0.3Wk, 0.5μW), uncertainty θ by LN(1.0, 0.05).

Parameters: χ=(Qk+Wk)/(Gk+Qk+Wk), k=Wk/Qk, factors γG, γQ,γW, ξ, ψQ, ψW.

1 Input data:

Normailised load effect $Ed := 1$

Range variables: $\chi := 0.001, 0.01 .. 0.99$ $\quad \gamma G := 1., 1.1 .. 1.9$ $\quad \gamma Q := 1.2, 1.25 .. 1.6$ $\quad \gamma W := 1.2, 1.25 .. 1.6$

Reduction factors: $\xi := 0.85$ $\quad \psi Q := 0.70$ $\quad \psi W := 0.60$ $\quad$ Loasd ratio: $k := 0.0$

2 Deterministic glabal factor:

Global load factor according to EN 1990 $\quad \gamma(\chi, \gamma G, \gamma Q, \gamma W) := \gamma G \cdot (1 - \chi) + (\gamma Q + k \cdot \gamma W) \cdot \frac{\chi}{(1 + k)}$

3 Probabilistic models

Permanent load G:	Normal distribution of G:		μG = Gk	$wG := 0.1$	
Variable load Q:	Gumbel distribution of Q:	μQ = mQ*Qk	$mQ := 0.6$	$wQ := 0.35$	$\alpha Q := 1.14$
Variable load W:	Gumbel distribution of W:	μW = mW*Wk	$mW := 0.3$	$wW := 0.5$	$\alpha W := 1.14$
Model uncertainty	Model uncertainty θ, Lognormal distribution:		$\mu\theta := 1.0$	$w\theta := 0.05$	$\alpha\theta := 3 \cdot w\theta + w\theta^3$

4 Determination of the load variables G, Q and W for a given load effect Ed = Rd

Characteristic values Gk=μG, Qk and Wk determined assuming Ed=Rd. Thus γG*Gk+γQ*Qk+γW*Wk=Rd

$$\mu G(\chi, \gamma G, \gamma Q, \gamma W) := \frac{Ed}{\left[\gamma G + \frac{(\gamma Q + k \cdot \gamma W) \cdot \chi}{(1 + k) \cdot (1 - \chi)}\right]}$$

Check: $\boxed{\mu G(0.4, 1.35, 1.5, 1.5) = 0.426}$

$$Qk(\chi, \gamma G, \gamma Q, \gamma W) := \frac{\chi \, \mu G(\chi, \gamma G, \gamma Q, \gamma W)}{(1 + k) \cdot (1 - \chi)} \qquad Wk(\chi, \gamma G, \gamma Q, \gamma W) := k \cdot Qk(\chi, \gamma G, \gamma Q, \gamma W)$$

$\boxed{Qk(0.40, 1.35, 1.5, 1.5) = 0.284}$

$$Ed(\chi, \gamma G, \gamma Q, \gamma W) := \gamma G \cdot \mu G(\chi, \gamma G, \gamma Q, \gamma W) + \gamma Q \cdot Qk(\chi, \gamma G, \gamma Q, \gamma W) + \gamma W \cdot Wk(\chi, \gamma G, \gamma Q, \gamma W)$$

$\boxed{Wk(0.5, 1.35, 1.5, 1.5) = 0}$

Normal distribution of G: $\sigma G(\chi, \gamma G, \gamma Q, \gamma W) := wG \cdot \mu G(\chi, \gamma G, \gamma Q, \gamma W)$ $\quad \boxed{Ed(0.5, 1.35, 1.5, 1.5) = 1}$

Gumbel distribution of Q: $\mu Q(\chi, \gamma G, \gamma Q, \gamma W) := mQ \cdot Qk(\chi, \gamma G, \gamma Q, \gamma W)$ $\quad \sigma Q(\chi, \gamma G, \gamma Q, \gamma W) := wQ \cdot \mu Q(\chi, \gamma G, \gamma Q, \gamma W)$

Gumbel distribution of W $\mu W(\chi, \gamma G, \gamma Q, \gamma W) := mW \cdot Wk(\chi, \gamma G, \gamma Q, \gamma W)$ $\quad \sigma W(\chi, \gamma G, \gamma Q, \gamma W) := wW \cdot \mu W(\chi, \gamma G, \gamma Q, \gamma W)$

5 Load effect E = θ * (G+Q+W) =θ*E0:

The mean and st. deviation of E0: $\mu E0(\chi, \gamma G, \gamma Q, \gamma W) := \mu G(\chi, \gamma G, \gamma Q, \gamma W) + \mu Q(\chi, \gamma G, \gamma Q, \gamma W) + \mu W(\chi, \gamma G, \gamma Q, \gamma W)$

$$\sigma E0(\chi, \gamma G, \gamma Q, \gamma W) := \sqrt{wG^2 \cdot \mu G(\chi, \gamma G, \gamma Q, \gamma W)^2 + wQ^2 \cdot \mu Q(\chi, \gamma G, \gamma Q, \gamma W)^2 + wW^2 \cdot \mu W(\chi, \gamma G, \gamma Q, \gamma W)^2}$$

The coefficient of variation of E0 (without model uncertainty θ):

$$\mu E0(0.4, 1.35, 1.5, 1.5) = 0.596$$

$$wE0(\chi, \gamma G, \gamma Q, \gamma W) := \frac{\sigma E0(\chi, \gamma G, \gamma Q, \gamma W)}{\mu E0(\chi, \gamma G, \gamma Q, \gamma W)} \quad \mu E(\chi, \gamma G, \gamma Q, \gamma W) := \mu\theta \cdot \mu E0(\chi, \gamma G, \gamma Q, \gamma W) \quad wE0(0.4, 1.35, 1.5, 1.5) = 0.123$$

The coefficient of variation of E: $wE(\chi, \gamma G, \gamma Q, \gamma W) := \sqrt{wE0(\chi, \gamma G, \gamma Q, \gamma W)^2 + w\theta^2 + wE0(\chi, \gamma G, \gamma Q, \gamma W)^2 \cdot w\theta^2}$

The standard deviation of E: $\sigma E(\chi, \gamma G, \gamma Q, \gamma W) := \mu E(\chi, \gamma G, \gamma Q, \gamma W) \cdot wE(\chi, \gamma G, \gamma Q, \gamma W)$

6 Three parameter lognormal distribution of E

$$wE(0.4, 1.35, 1.5, 1.5) = 0.133$$

Skewness of E0:

$$\alpha E0(\chi, \gamma G, \gamma Q, \gamma W) := \frac{\sigma Q(\chi, \gamma G, \gamma Q, \gamma W)^3 \cdot \alpha Q + \sigma W(\chi, \gamma G, \gamma Q, \gamma W)^3 \cdot \alpha W}{\sigma E0(\chi, \gamma G, \gamma Q, \gamma W)^3}$$

$$\boxed{\alpha E0(0.4, 1.35, 1.5, 1.5) = 0.614}$$

Skewness of E:

$$\alpha E(\chi, \gamma G, \gamma Q, \gamma W) := \frac{wE0(\chi, \gamma G, \gamma Q, \gamma W)^3 \cdot \alpha E0(\chi, \gamma G, \gamma Q, \gamma W) + 6 \cdot w\theta^2 \cdot wE0(\chi, \gamma G, \gamma Q, \gamma W)^2 + w\theta^3 \cdot \alpha\theta}{wE(\chi, \gamma G, \gamma Q, \gamma W)^3}$$

$$\boxed{\alpha E(0.4, 1.35, 1.5, 1.5) = 0.591}$$

Parameter C:

$$C(\chi, \gamma G, \gamma Q, \gamma W) := \frac{\left(\sqrt{\alpha E(\chi, \gamma G, \gamma Q, \gamma W)^2 + 4} + \alpha E(\chi, \gamma G, \gamma Q, \gamma W)\right)^{\frac{1}{3}} - \left(\sqrt{\alpha E(\chi, \gamma G, \gamma Q, \gamma W)^2 + 4} - \alpha E(\chi, \gamma G, \gamma Q, \gamma W)\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}}$$

Parameters of transformed variable:

$$mE(\chi, \gamma G, \gamma Q, \gamma W) := -\ln(|C(\chi, \gamma G, \gamma Q, \gamma W)|) + \ln(\sigma E(\chi, \gamma G, \gamma Q, \gamma W)) - (0.5) \cdot \ln\left(1 + C(\chi, \gamma G, \gamma Q, \gamma W)^2\right)$$

$$sE(\chi, \gamma G, \gamma Q, \gamma W) := \sqrt{\ln\left(1 + C(\chi, \gamma G, \gamma Q, \gamma W)^2\right)} \quad x0(\chi, \gamma G, \gamma Q, \gamma W) := \mu E(\chi, \gamma G, \gamma Q, \gamma W) - \frac{1}{C(\chi, \gamma G, \gamma Q, \gamma W)} \sigma E(\chi, \gamma G, \gamma Q, \gamma W)$$

Probability density of E, approximation by three parameter lognormal distribution: $\boxed{x0(0.4, 1.35, 1.5, 1.5) = 0.189}$

$$Eln(x, \chi, \gamma G, \gamma Q, \gamma W) := dlnorm(x - x0(\chi, \gamma G, \gamma Q, \gamma W), mE(\chi, \gamma G, \gamma Q, \gamma W), sE(\chi, \gamma G, \gamma Q, \gamma W))$$

7 Theoretical value of the global factor γ for a given exceedance probability p of Ed:

p = P{ E > Ed } $p := 0.001, 0.0011 .. 0.006$ Probability considered in EN: $\boxed{1 - pnorm(0.7 \cdot 3.8, 0, 1) = 3.907 \times 10^{-3}}$

$$pp(Ed, \chi, \gamma G, \gamma Q, \gamma W) := plnorm(Ed - x0(\chi, \gamma G, \gamma Q, \gamma W), mE(\chi, \gamma G, \gamma Q, \gamma W), sE(\chi, \gamma G, \gamma Q, \gamma W))$$

$$Ed(p, \chi, \gamma G, \gamma Q, \gamma W) := x0(\chi, \gamma G, \gamma Q, \gamma W) + qlnorm(1 - p, mE(\chi, \gamma G, \gamma Q, \gamma W), sE(\chi, \gamma G, \gamma Q, \gamma W))$$

$$Ek(\chi, \gamma G, \gamma Q, \gamma W) := \mu G(\chi, \gamma G, \gamma Q, \gamma W) + Qk(\chi, \gamma G, \gamma Q, \gamma W) + Wk(\chi, \gamma G, \gamma Q, \gamma W)$$

$$\gamma p(p, \chi, \gamma G, \gamma Q, \gamma W) := \frac{Ed(p, \chi, \gamma G, \gamma Q, \gamma W)}{Ek(\chi, \gamma G, \gamma Q, \gamma W)}$$

$$\boxed{pp(0.8, 0.4, 1.35, 1.5, 1.5) = 0.986}$$

$$\boxed{Ed(0.002, 0.4, 1.35, 1.5, 1.5) = 0.884}$$

$$\boxed{\gamma p(0.002, 0.3, 1.35, 1.5, 1.5) = 1.228}$$

$$\boxed{\gamma p(0.002, 0.3, 1.5, 1.5, 1.5) = 1.228}$$

8 The global load factor γ versus ratio χ:

limit for dominant action: $k0 := \frac{1-\psi Q}{1-\psi W}$ Check: $\boxed{k0 = 0.75}$

Auxiliary quantities: $a := \mathrm{if}(k \le k0, 1, \psi Q)$ $b := \mathrm{if}(k > k0, 1, \psi W)$ $\boxed{a = 1}$ $\boxed{b = 0.6}$

Limit value of χ for (6.10a) and (6.10b) $\chi\chi(\gamma G, \gamma Q, \gamma W) := \frac{\gamma G \cdot (1-\xi)(1+k)}{\gamma G \cdot (1-\xi)(1+k) + [\gamma Q \cdot (a-\psi Q) + \gamma W \cdot k \cdot (b-\psi W)]}$

Limit of χ for (6.10a-mod) and (6.10b) $\chi\chi a(\gamma G, \gamma Q, \gamma W) := \frac{\gamma G \cdot (1-\xi)(1+k)}{\gamma G \cdot (1-\xi)(1+k) + (\gamma Q \cdot a + \gamma W \cdot k \cdot b)}$ $\chi la := \chi\chi a(1.4, 1.6, 1.6)$

$\chi l := \chi\chi(1.35, 1.5, 1.5)$

Target probability $\beta t := 3.8$ Auxiliary: $\chi 0 := 1, 1.1 .. 1.3$

$\chi a := 0, 0.05 .. \chi l + 0.05$ $\chi b := \chi la - 0.01, \chi la + 0.04 .. 0.999$ $\chi c := 0, 0.05 .. \chi la + 0.04$ Check: $\boxed{\chi l = 0.31}$

Turkstra's for 50 years: $wG = 0.1$ $mQ = 0.6$ $wQ = 0.35$ $mW = 0.3$ $wW = 0.5$ $k = 0$ $\boxed{\chi la = 0.116}$

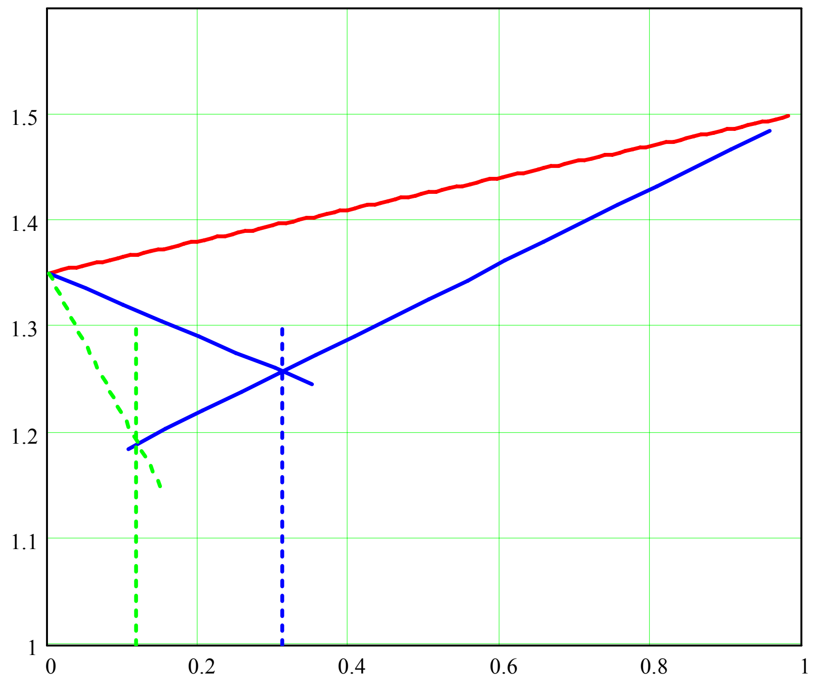


Figure 1: Deterministic global load factor γE versus χ, code values corresponding to equation (6.10), (6.10a) and (6.10b).

The global load factor γE corresponding to expression (6.10) - combination A is represented by solid red line, factor corresponding to expressions (6.10a) and (6.10b) - combination B by blue line and factor corresponding to expression (6.10a,mod) - combination C by dashed green line. Vertical dashed lines indicate boundaries for validity of expressions (6.10a) and (6.10b) or expressions (6.10a,mod) and (6.10b).

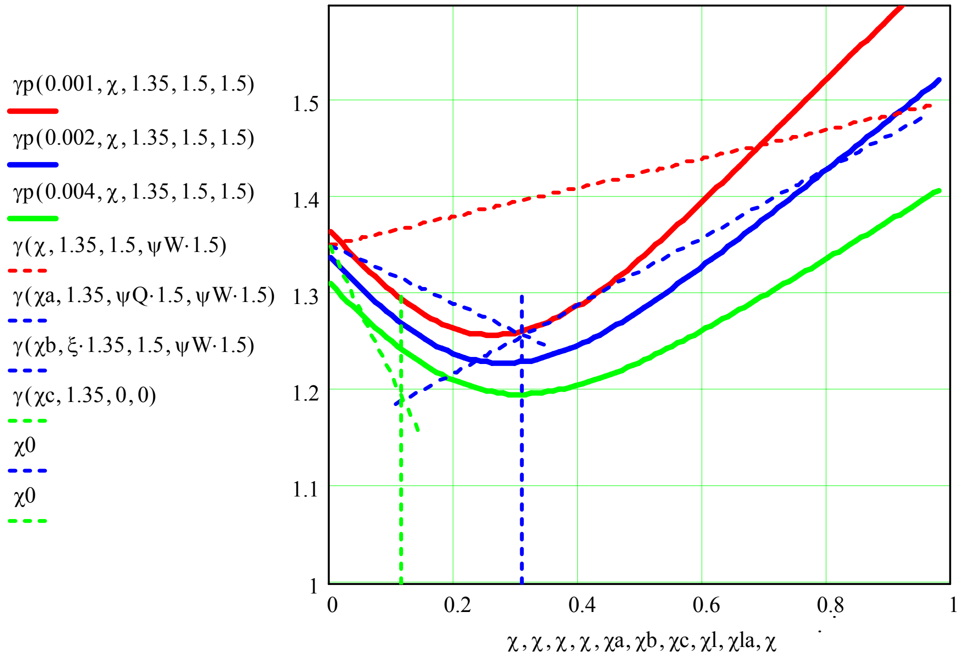


Figure 2: Global load factor γ versus χ, theoretical and deterministic values corresponding to equation (6.10), (6.10a) and (6.10b).

9 Global index γ versus χ and γQ

$\gamma G := 1.35 \quad \gamma Q := 1.5 \quad \gamma W := 1.5 \quad \Gamma(p,\chi) := \gamma p(p,\chi,\gamma G, a\cdot\gamma Q, b\cdot\gamma W)$

$\gamma\gamma(p,\chi) := \gamma(\chi, 1\cdot 1.35, 1\cdot 1.5, \psi W\cdot 1.5)$

Parameters : $k = 0$

$mQ = 0.6 \quad wQ = 0.35$

$mW = 0.3 \quad wW = 0.5$

$a = 1 \quad b = 0.6$

$\Gamma(0.004, 0.35) = 1.198$

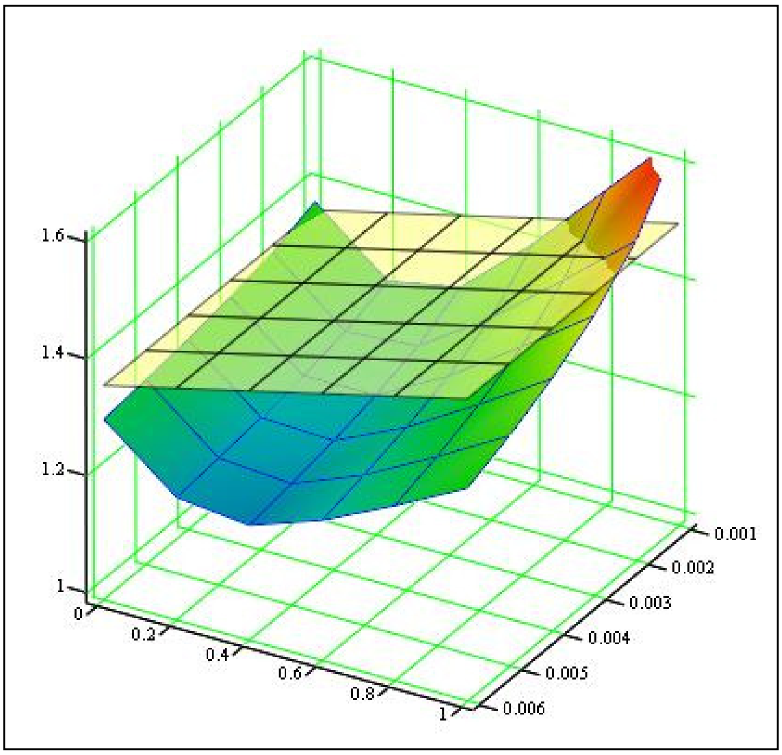


Figure 3: Global factor γ versus probability p and load ratio χ for load combination given by equation (6.10).

$\Gamma, \gamma\gamma$